Asymptotics of smoothed Wasserstein distances in the small noise regime
Supplementary Material

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Contents

1 Omitted proofs for Section 3

1.1 Proof of Proposition 3.6

Proof. Suppose Γ is f-strongly cyclically monotone for some positive residual f. Denote

$$M := \max\left\{\max_i \|x_i\|, \max_i \|y_i\|\right\}.$$

We will show that Γ is ϵ-robust for any $\epsilon > 0$ satisfying

$$4M\epsilon < \min_{i\neq j} f(i,j).$$

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In fact, for any distinct $\tau(1), \tau(2), \ldots, \tau(n) \in [k]$, by the definition of f-strong cyclical monotonicity,

$$\sum_{i=1}^{n} \langle x_{\tau(i)}, y_{\tau(i)} - y_{\tau(i+1)} \rangle \geq \sum_{i=1}^{n} f(\tau(i), \tau(i+1))$$

Thus for any choice of $\alpha_{\tau(1)}, \ldots, \alpha_{\tau(n)}$ such that $\max \|\alpha_{\tau(i)}\| \leq \epsilon$, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n} \|(x_{\tau(i)} + \alpha_{\tau(i)}) - (y_{\tau(i+1)} + \alpha_{\tau(i+1)})\|^2 - \frac{1}{2}\sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i)}\|^2 \\
&= \sum_{i=1}^{n} \langle x_{\tau(i)}, y_{\tau(i)} - y_{\tau(i+1)} \rangle + \sum_{i=1}^{n} \langle \alpha_{\tau(i)}, x_{\tau(i)} - x_{\tau(i-1)} + y_{\tau(i)} - y_{\tau(i+1)} \rangle + \frac{1}{2}\sum_{i=1}^{n} \|\alpha_{\tau(i)} - \alpha_{\tau(i+1)}\|^2 \\
&\geq \sum_{i=1}^{n} f(\tau(i), \tau(i+1)) - 4nM\epsilon \\
&> 0.
\end{aligned}$$

Hence $R(\Gamma) > 0$.

On the other hand, given $R(\Gamma) > 0$, we show that Γ is the unique optimal transport plan from $\{x_i\}$ to $\{y_i\}$. We prove by contradiction. If Γ is not unique, then there exists distinct $\tau(1), \ldots, \tau(n) \in [k]$ such that

$$\sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i)}\|^2 = \sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i+1)}\|^2. \tag{1}$$

Since $R(\Gamma) > 0$, for $\epsilon_0 = R(\Gamma)/2$ and any choice of $\tau(1), \ldots, \tau(n)$ with $\|\tau(i)\| \leq \epsilon_0$, we have

$$\sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i)}\|^2 \leq \sum_{i=1}^{n} \|(x_{\tau(i)} + \alpha_{\tau(i)}) - (y_{\tau(i+1)} + \alpha_{\tau(i+1)})\|^2.$$

Specifically, for any $j \in [n]$, letting $\tau(i) = 0$ for all $i \neq j$ in the above equation gives

$$2\langle \alpha_{\tau(j)}, x_{\tau(j)} - y_{\tau(j+1)} \rangle \leq \|\alpha_{\tau(j)}\|^2$$

for any $\alpha_{\tau(j)} \in \mathbb{R}^d$ with $\|\alpha_{\tau(j)}\| \leq \epsilon_0$. Therefore we must have

$$x_{\tau(j)} = y_{\tau(j+1)}, \quad \forall\, j \in [k].$$

Using (1), we also know that

$$x_{\tau(j)} = y_{\tau(j)}, \quad \forall\, j \in [k],$$

which violates the assumption that $\{y_i\}$ are distinct points in $\mathbb{R}^d$. Thus we conclude that Γ is unique; hence it is also strongly cyclically monotone due to Proposition 3.8. ☐

1.2 Proof of Proposition 3.8

Proof. (i) to (ii). The idea is borrowed from [1, 2, 3]. Suppose Γ is f-strongly cyclically monotone for a positive residual function f. For $i \in [k]$, denote

$$v_i := \inf_{\substack{\theta(1)=1,\theta(n+1)=i,\\ \theta(2),\ldots,\theta(n)\in[k],\\ \theta(s)\neq\theta(s+1)}} \left(\sum_{s=1}^{n} \langle x_{\theta(s)}, y_{\theta(s)} - y_{\theta(s+1)} \rangle - \sum_{s=1}^{n} f(\theta(s), \theta(s+1)) \right)$$

By the f-strong cyclical monotonicity, we have $v_1 \geq 0$. Furthermore, for $i > 1$ and any sequence $\{\theta(s)\}$ with $\theta(1) = 1$, $\theta(n+1) = i$ and $\theta(s) \neq \theta(s+1)$, there holds

$$\sum_{s=1}^{n} \langle x_{\theta(s)}, y_{\theta(s)} - y_{\theta(s+1)} \rangle + \langle x_i, y_i - y_1 \rangle \geq \sum_{s=1}^{n} f(\theta(s), \theta(s+1)) + f(i, 1)$$

and it follows that

$$v_i \geq f(i, 1) - \langle x_i, y_i - y_1 \rangle > -\infty.$$

For any $j \neq i$ and any fixed $\epsilon > 0$, there exists a sequence $\{\theta(s)\}$ with $\theta(1) = 1$, $\theta(n+1) = i$ and $\theta(s) \neq \theta(s+1)$, such that

$$\sum_{s=1}^{n} \langle x_{\theta(s)}, y_{\theta(s)} - y_{\theta(s+1)} \rangle - \sum_{s=1}^{n} f(\theta(s), \theta(s+1)) \leq v_i + \epsilon. \tag{2}$$

Consider the same $\{\theta(s)\}$ with one more term $\theta(n+2) := j$. By definition of v_j we have

$$v_j \leq \sum_{s=1}^{n} \langle x_{\theta(s)}, y_{\theta(s)} - y_{\theta(s+1)} \rangle + \langle x_i, y_i - y_j \rangle - \sum_{s=1}^{n+1} f(\theta(s), \theta(s+1)) \tag{3}$$

Comparing (2) and (3) we get

$$v_j \leq v_i + \langle x_i, y_i - y_j \rangle - f(i,j) + \epsilon \tag{4}$$

We set $\varphi(x_i) = -v_i$. Letting $\epsilon \downarrow 0$ in (4) yields

$$\langle x_i, y_i - y_j \rangle \geq \varphi(x_i) - \varphi(x_j) + f(i,j).$$

Hence Γ is f-strongly implementable.
(ii) to (iii). We prove by contradiction. Suppose Γ is not the unique optimal transport plan; this means either Γ is not optimal or there exists a different coupling Γ' with the same cost. Either case, there exists a sequence $\{\theta(s)\}_{s=1}^n$ such that

$$\sum_{s=1}^{n} \|x_{\theta(s)} - y_{\theta(s)}\|^2 \geq \sum_{s=1}^{n} \|x_{\theta(s)} - y_{\theta(s+1)}\|^2$$

Summing over s, we get

$$\begin{aligned}
\sum_{s=1}^{n} f(\theta(s), \theta(s+1)) &\leq \sum_{s=1}^{n} \langle x_{\theta(s)}, y_{\theta(s)} - y_{\theta(s+1)} \rangle \\
&= \frac{1}{2} \left(\sum_{s=1}^{n} \|x_{\theta(s)} - y_{\theta(s+1)}\|^2 - \sum_{s=1}^{n} \|x_{\theta(s)} - y_{\theta(s)}\|^2 \right) \\
&\leq 0,
\end{aligned}$$

a contradiction.
(iii) to (i). Suppose Γ is the unique optimal transport plan from $\{x_i\}$ to $\{y_i\}$. Denote c_0 the transport cost of Γ. For any other transport plan in the form of a bijection between $\{x_i\}$ and $\{y_i\}$, denote c_1 the minimum among their costs, then $c_1 > c_0$. Choose a small enough $\lambda > 0$, such that for any choice of $\tau(1), \tau(2), \ldots, \tau(n) \in [k]$ with no duplicates, there holds

$$\frac{\lambda}{2} \sum_{i=1}^{n} \|y_{\tau(i)} - y_{\tau(i+1)}\|^2 \leq c_1 - c_0.$$

Now for $f(i,j) = \frac{\lambda}{2}\|y_i - y_j\|^2$ we have

$$\sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i+1)}\|^2 - \sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i)}\|^2 \geq c_1 - c_0 \geq \sum_{i=1}^{n} f(\tau(i), \tau(i+1)).$$

If there are duplicates in $(\tau(1), \tau(2), \ldots, \tau(n))$, we break the loop $\tau(1) \to \tau(2) \to \cdots \to \tau(n) \to \tau(1)$ into separate loops without duplicates, apply the above inequality to each loop and sum them up. We conclude by definition that Γ is f-strongly cyclically monotone. □

1.3 Proof of Proposition 3.13

Proof of Proposition 3.13. We only need to show that, for an ϵ satisfying (7), and any choice of $\tau(1), \tau(2), \ldots, \tau(n) \in [k]$, and $\alpha(1), \ldots, \alpha(n)$ with $\|\alpha(i)\| \leq \epsilon$, there holds

$$\sum_{i} \|x_{\tau(i)} - y_{\tau(i)}\|^2 \leq \sum_{i} \|(x_{\tau(i)} + \alpha_{\tau(i)}) - (y_{\tau(i+1)} + \alpha_{\tau(i+1)})\|^2. \tag{5}$$

In fact, (5) is equivalent to

$$2\sum_i \langle \alpha_{\tau(i)}, y_{\tau(i+1)} - y_{\tau(i)} + x_{\tau(i-1)} - x_{\tau(i)} \rangle \le 2\sum_i \langle x_{\tau(i)}, y_{\tau(i)} - y_{\tau(i+1)} \rangle + \sum_i \|\alpha_{\tau(i)} - \alpha_{\tau(i+1)}\|^2 \tag{6}$$

Since $\|\alpha(i)\| \le \epsilon$ for all i, we have

$$\begin{aligned}
&2\sum_i \langle \alpha_{\tau(i)}, y_{\tau(i+1)} - y_{\tau(i)} + x_{\tau(i-1)} - x_{\tau(i)} \rangle \\
&\le 2\sum_i \epsilon \cdot \left(\|y_{\tau(i+1)} - y_{\tau(i)}\| + \|x_{\tau(i+1)} - x_{\tau(i)}\| \right) \\
&\le \sum_i f(\tau(i), \tau(i+1))
\end{aligned}$$

where we used the choice of ϵ in the last inequality. In the meantime, strong implementability gives

$$2\sum_i \langle x_{\tau(i)}, y_{\tau(i)} - y_{\tau(i+1)} \rangle + \sum_i \|\alpha_{\tau(i)} - \alpha_{\tau(i+1)}\|^2 \ge \sum_i f(\tau(i), \tau(i+1)).$$

Therefore (6) holds, which completes the proof. □

1.4 Proof of Proposition 3.14

Proof. Following the proof of Proposition 3.13, we only need to show that, for the residual $f(i, j)$ defined in Theorem 3.10, there holds

$$2\sum_i \epsilon \cdot \left(\|y_{\tau(i+1)} - y_{\tau(i)}\| + \|x_{\tau(i+1)} - x_{\tau(i)}\| \right) \le \sum_i f(\tau(i), \tau(i+1)). \tag{7}$$

By the choice of ϵ, we have

$$\begin{aligned}
&2\sum_i \epsilon \cdot \left(\|y_{\tau(i+1)} - y_{\tau(i)}\| + \|x_{\tau(i+1)} - x_{\tau(i)}\| \right) \\
&\le \sum_i \max\left\{ \frac{1}{\beta}\|x_{\tau(i+1)} - x_{\tau(i)}\|^2, \alpha\|y_{\tau(i+1)} - y_{\tau(i)}\|^2 \right\}.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
&\sum_i f(\tau(i), \tau(i+1)) \\
&= \frac{1}{\beta - \alpha} \sum_i \left(\|x_{\tau(i)} - x_{\tau(i+1)}\|^2 + \alpha\beta\|y_{\tau(i)} - y_{\tau(i+1)}\|^2 - 2\alpha\langle y_{\tau(i)} - y_{\tau(i+1)}, x_{\tau(i)} - x_{\tau(i+1)} \rangle \right) \\
&\ge \frac{1}{\beta - \alpha} \sum_i \left(\|x_{\tau(i)} - x_{\tau(i+1)}\|^2 + \alpha\beta\|y_{\tau(i)} - y_{\tau(i+1)}\|^2 - \alpha\left(\lambda\|x_{\tau(i)} - x_{\tau(i+1)}\|^2 + \frac{1}{\lambda}\|y_{\tau(i)} - y_{\tau(i+1)}\|^2 \right) \right).
\end{aligned}$$

The last inequality holds for any $\lambda > 0$ by the Cauchy-Schwarz inequality. Choosing $\lambda = 1/\beta$ and $\lambda = 1/\alpha$ yields

$$\sum_i f(\tau(i), \tau(i+1)) \ge \max\left\{ \frac{1}{\beta}\|x_{\tau(i+1)} - x_{\tau(i)}\|^2, \alpha\|y_{\tau(i+1)} - y_{\tau(i)}\|^2 \right\}.$$

Therefore (7) holds, which completes the proof. □

2 Omitted proofs for Section 4

2.1 Proof of Theorem 4.1

Proof. Define the truncated smoothing kernel

$$\tilde{\mathcal{N}}_\sigma := \mathcal{N}(0, \sigma^2 I) \cdot \mathbf{1}\{\|X\| \le \epsilon_*\} + (1-p)\delta_0$$

where

$$p = \mathbb{P}\left[\|\mathcal{N}(0,\sigma^2 I)\| < \epsilon_*\right].$$

Since $\tilde{\mathcal{N}}_\sigma$ is supported on $B(0,\epsilon_*)$, by Lemma 4.2, we know

$$W_2(\mu * \tilde{\mathcal{N}}_\sigma, \nu * \tilde{\mathcal{N}}_\sigma) = W_2(\mu,\nu).$$

Therefore,

$$\begin{aligned}
&|W_2(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) - W_2(\mu,\nu)|^2 \\
&= |W_2(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) - W_2(\mu * \tilde{\mathcal{N}}_\sigma, \nu * \tilde{\mathcal{N}}_\sigma)|^2 \\
&\le (W_2(\mu * \mathcal{N}_\sigma, \mu * \tilde{\mathcal{N}}_\sigma) + W_2(\nu * \mathcal{N}_\sigma, \nu * \tilde{\mathcal{N}}_\sigma))^2 \\
&\lesssim \mathbb{E}_{z\sim\mathcal{N}(0,\sigma^2 I)}\left[\|z\|^2 \mathbf{1}_{\|z\|\ge\sigma_*}\right] \\
&= \sigma^2\, \mathbb{E}_{z\sim\mathcal{N}(0,I)}\left[\|z\|^2 \mathbf{1}_{\|z\|\ge\sigma_*/\sigma}\right] \\
&\lesssim \sigma\sigma_* e^{-\sigma_*^2/2\sigma^2}.
\end{aligned}$$

Here the second inequality is yielded by considering a coupling of $\mu * \mathcal{N}_\sigma$ and $\mu * \tilde{\mathcal{N}}_\sigma$ that is the distribution of $(X+Z, X+Z\cdot\mathbf{1}\{\|Z\|\le\epsilon_*\})$, where X and Z are independent, $X\sim\mu$ and $Z\sim\mathcal{N}(0,\sigma^2 I)$, and the same coupling for μ replaced with ν. Taking square root on both sides yields the result. □

2.2 Proof of Lemma 4.2

Proof. We naturally split the source measure into k parts:

$$\mu * Q = \sum_{i=1}^k \left(\frac{1}{k}\delta(x_i) * Q\right)$$

Consider a map T which, for each $i \in [k]$, is defined by

$$T(x) = x + y_i - x_i \qquad \forall x \in B(x_i, \sigma_*).$$

We can obtain a transport plan between $\mu * Q$ and $\nu * Q$ by considering the distribution of a pair of random variables $(X, T(X))$ for $X \sim \mu * Q$. The support of this plan lies in the set $\bigcup_{i=1}^k \bigcup_{\alpha\in B(0,\sigma_*)} (x_i+\alpha, y_i+\alpha)$. By the definition of $R(\Gamma)$, this set is cyclically monotone, so this coupling is optimal for $\mu * Q$ and $\nu * Q$ by Theorem 3.2. Therefore

$$\begin{aligned}
W_2^2(\mu * Q, \nu * Q) &= \int \|x - T(x)\|^2 d(\mu * Q)(x) \\
&= \frac{1}{k}\sum_{i=1}^k \|y_i - x_i\|^2 = W_2^2(\mu,\nu),
\end{aligned}$$

as claimed. □

2.3 Proof of Proposition 4.3

Proof. For $M > 0$, denote

$$g(m) := \sup\left\{\sum_{i=1}^n \|x_{\tau(i)} - y_{\tau(i)}\|^2 - \sum_{i=1}^n \|(x_{\tau(i)} + \alpha_{\tau(i)}) - (y_{\tau(i+1)} + \alpha_{\tau(i+1)})\|^2 :\ \max_i \|\alpha_{\tau(i)}\| = m\right\},$$

then $G(M) = \sup\{g(m) : m \in [0, M]\}$. We first prove that $g(m)$ is concave in m. In fact, denote the set

$$\mathcal{I} = \left\{(\tau(1),\ldots,\tau(n),\alpha_{\tau(1)},\ldots,\alpha_{\tau(n)}) :\ \tau(i)\in[k],\ \tau(i)\ne\tau(j),\ \max_i \|\alpha_{\tau(i)}\| = 1\right\}.$$

By definition,

$$\begin{aligned}
g(m) = \sup\Bigg\{&\sum_{i=1}^n \|x_{\tau(i)} - y_{\tau(i)}\|^2 - \sum_{i=1}^n \|(x_{\tau(i)} + m\alpha_{\tau(i)}) - (y_{\tau(i+1)} + m\alpha_{\tau(i+1)})\|^2 : \\
&(\tau(1),\ldots,\tau(n),\alpha_{\tau(1)},\ldots,\alpha_{\tau(n)}) \in \mathcal{I}\Big\}
\end{aligned}$$

Note that, for every choice of $(\tau(1), \dots, \tau(n))$ and $\alpha_{\tau(1)}, \dots, \alpha_{\tau(n)}) \in \mathcal{I}$,

$$\sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i)}\|^2 - \sum_{i=1}^{n} \|(x_{\tau(i)} + m\alpha_{\tau(i)}) - (y_{\tau(i+1)} + m\alpha_{\tau(i+1)})\|^2$$

is a concave function in m. Therefore, $g(m)$ is concave in m, and $G(M)$ is also concave in M. $\square$

2.4 Proof of Theorem 4.4

Proof. For $M > \sigma_*$, pick $\tau(1), \tau(2), \dots, \tau(n) \in [k]$ and $\{\alpha_{\tau(i)}\}_{i=1}^n \subset \mathbb{R}^d$ such that $\|\alpha_{\tau(i)}\| \le M$ and

$$\sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i)}\|^2 - \sum_{i=1}^{n} \|(x_{\tau(i)} + \alpha_{\tau(i)}) - (y_{\tau(i+1)} + \alpha_{\tau(i+1)})\|^2 = G(M).$$

For every $i \in [k]$, denote $B_{\tau(i)}$ the ball centered at $x_{\tau(i)} + \alpha_{\tau(i)}$ with radius σ, and $\hat{B}_{\tau(i)}$ the ball centered at $y_{\tau(i)} + \alpha_{\tau(i)}$ with radius σ. Also denote

- $\gamma \in \Pi(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma)$ the law of $(X+Z, Y+Z)$, where $(X,Y) \sim \frac{1}{k}\sum_{i=1}^k \delta(x_i, y_i)$ and $Z \sim \mathcal{N}_\sigma$ are independent.
- $\gamma_{\tau(i)} \in \Pi(\mathsf{Unif}(B_{\tau(i)}), \mathsf{Unif}(\hat{B}_{\tau(i)}))$ the coupling associated with the transport map
$$x \mapsto x + y_{\tau(i)} - x_{\tau(i)};$$
- $\tilde{\gamma}_{\tau(i)} \in \Pi(\mathsf{Unif}(B_{\tau(i)}), \mathsf{Unif}(\hat{B}_{\tau(i+1)}))$ the coupling associated with the transport map
$$x \mapsto x + y_{\tau(i+1)} - x_{\tau(i)};$$
- A constant $m = c_d \exp\left(-\frac{(M+\sigma)^2}{2\sigma^2}\right)$, where c_d is a constant only dependent on the dimension d.

Consider the following measure in $\mathbb{R}^d \times \mathbb{R}^d$:

$$\tilde{\gamma} := \gamma - m\sum_{i=1}^{n} \gamma_{\tau(i)} + m\sum_{i=1}^{n} \tilde{\gamma}_{\tau(i)}.$$

We shall show that $\tilde{\gamma} \in \Pi(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma)$. We first verify that $\tilde{\gamma}$ is a positive measure on $\mathbb{R}^d \times \mathbb{R}^d$. In fact, for $x, y \in \mathbb{R}^d$,

$$\gamma(dx, dy) = \frac{1}{k}\sum_{i=1}^{k} \left(\frac{1}{(\sqrt{2\pi}\sigma)^d} e^{-\frac{\|x-x_i\|^2}{2\sigma^2}} dx \cdot \delta_{x-x_i+y_i}(dy) \right).$$

Meanwhile,

$$\left(m\sum_{i=1}^{n} \gamma_{\tau(i)} \right)(dx, dy) = m\sum_{i=1}^{n} \left(\frac{\mathbf{1}\{x \in B_{\tau(i)}\}}{\mathsf{Vol}(B_{\tau(i)})} dx \cdot \delta_{x-x_{\tau(i)}+y_{\tau(i)}}(dy) \right).$$

For every $\tau(i)$ such that $x \in B_{\tau(i)}$, note that

$$\|x - x_{\tau(i)}\| \le \|x - (x_{\tau(i)} + \alpha_{\tau(i)})\| + \|\alpha_{\tau(i)}\| \le \sigma + M,$$

hence (with a proper choice of c_d)

$$\frac{1}{k}\frac{1}{(\sqrt{2\pi}\sigma)^d} e^{-\frac{\|x-x_{\tau(i)}\|^2}{2\sigma^2}} \ge \frac{1}{k}\frac{1}{(\sqrt{2\pi}\sigma)^d} e^{-\frac{(M+\sigma)^2}{2\sigma^2}} \ge \frac{m}{\mathsf{Vol}(B_{\tau(i)})}.$$

As a result, $\gamma - m\sum_{i=1}^n \gamma_{\tau(i)} \ge 0$, and $\tilde{\gamma}$ is a positive measure. Also note that its first marginal (i.e. the marginal on the first d dimensions) and second marginal (i.e. the marginal on the last d

dimensions) agree with the respective marginals of γ. Thus we conclude that $\tilde{\gamma} \in \Pi(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma)$. Now note that

$$\begin{aligned} &\int c(x,y) d\gamma(x,y) - \int c(x,y) d\tilde{\gamma}(x,y) \\ &= m\left(\sum_{i=1}^{n} \|x_{\tau(i)} - y_{\tau(i)}\|^2 - \sum_{i=1}^{n} \|(x_{\tau(i)} + \alpha_{\tau(i)}) - (y_{\tau(i+1)} + \alpha_{\tau(i+1)})\|^2\right) \\ &= m \cdot G(M). \end{aligned}$$

In the meantime,

$$\int c(x,y) d\gamma(x,y) = \frac{1}{2k} \sum_{i=1}^{k} \|x_i - y_i\|^2 = W_2^2(\mu, \nu),$$

therefore,

$$\begin{aligned} &W_2^2(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) \\ &\leq \int c(x,y) d\tilde{\gamma}(x,y) \\ &\leq W_2^2(\mu, \nu) - G(M) \cdot c_d \exp\left(-\frac{(M+\sigma)^2}{2\sigma^2}\right). \end{aligned}$$

In particular, choosing $M = \sigma + \sigma_*$ yields

$$W_2^2(\mu, \nu) - W_2^2(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) \gtrsim G(\sigma + \sigma_*) \exp\left(-c\frac{\sigma_*^2}{\sigma^2}\right).$$

The rest follows from the observation that, for $\sigma \in (0, 2\sigma_*)$,

$$G(\sigma + \sigma_*) = G(\sigma + \sigma_*) - G(\sigma_*) \geq \frac{G(3\sigma_*) - G(\sigma_*)}{2\sigma_*} \cdot \sigma$$

since G is concave by Proposition 4.3. □

3 Omitted proofs for Section 5

3.1 Proof of Theorem 5.1

Proof. Suppose that there exists a transport plan π between μ and ν which achieves the optimal cost and is not a perfect matching. Without loss of generality, we assume that (x_1, y_1) and (x_1, y_2) both lie in the support of π. Let $\lambda = \min\{\pi(x_1, y_1), \pi(x_1, y_2)\}$. We decompose μ and ν as

$$\begin{aligned} \hat{\mu} = \mu - 2\lambda\delta(x_1), \quad \tilde{\mu} = 2\lambda\delta(x_1), \\ \hat{\nu} = \nu - \lambda\left(\delta(y_1) + \delta(y_2)\right), \quad \tilde{\nu} = \lambda\left(\delta(y_1) + \delta(y_2)\right). \end{aligned}$$

By Lemma 5.2, there exists $c_0 > 0$ such that for $\sigma \in (0, c_0)$,

$$W_2^2(\tilde{\mu}, \tilde{\nu}) - W_2^2(\tilde{\mu} * \mathcal{N}_\sigma, \tilde{\nu} * \mathcal{N}_\sigma) \gtrsim \sigma.$$

Therefore, for $\sigma \in (0, c_0)$, we also have

$$\begin{aligned} &W_2^2(\mu, \nu) - W_2^2(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) \\ &\geq W_2^2(\hat{\mu}, \hat{\nu}) - W_2^2(\hat{\mu} * \mathcal{N}_\sigma, \hat{\nu} * \mathcal{N}_\sigma) + W_2^2(\tilde{\mu}, \tilde{\nu}) - W_2^2(\tilde{\mu} * \mathcal{N}_\sigma, \tilde{\nu} * \mathcal{N}_\sigma) \\ &\geq W_2^2(\tilde{\mu}, \tilde{\nu}) - W_2^2(\tilde{\mu} * \mathcal{N}_\sigma, \tilde{\nu} * \mathcal{N}_\sigma) \\ &\gtrsim \sigma. \end{aligned}$$

□

3.2 Proof of Lemma 5.2

Proof. First suppose that x, y_1, y_2 are not on the same line with y_1 between x and y_2 or y_2 between x and y_1. Let Δ be the bisecting hyperplane of $\angle y_1 x y_2$, namely

$$\Delta = \left\{ z \in \mathbb{R}^d \ : \ \frac{\langle z - x, y_1 - x \rangle}{|y_1 - x|} = \frac{\langle z - x, y_2 - x \rangle}{|y_2 - x|} \right\},$$

and define its unit normal vector $\mathbf{m}$ such that $\langle \mathbf{m}, y_1 - x \rangle > 0$. We adopt the decomposition

$$\begin{aligned} \mu_+ &:= \mathcal{N}(x, \sigma^2) \mid \langle z - x, \mathbf{m} \rangle > 0, \\ \mu_- &:= \mathcal{N}(x, \sigma^2) \mid \langle z - x, \mathbf{m} \rangle < 0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \nu_{1+} &:= \mathcal{N}(y_1, \sigma^2) \mid \langle z - y_1, \mathbf{m} \rangle > 0, \\ \nu_{1-} &:= \mathcal{N}(y_1, \sigma^2) \mid \langle z - y_1, \mathbf{m} \rangle < 0, \\ \nu_{2+} &:= \mathcal{N}(y_2, \sigma^2) \mid \langle z - y_2, \mathbf{m} \rangle > 0, \\ \nu_{2-} &:= \mathcal{N}(y_2, \sigma^2) \mid \langle z - y_2, \mathbf{m} \rangle < 0. \end{aligned} \tag{9}$$

Note that all the six sub-probability measures above have mass $1/2$. By the definition of W_2, we have

$$W_2^2(\mu_0 * \mathcal{N}_\sigma, \nu_0 * \mathcal{N}_\sigma) \leq \frac{1}{2} \left(W_2^2(\mu_+, \nu_{1+}) + W_2^2(\mu_+, \nu_{1-}) + W_2^2(\mu_-, \nu_{2+}) + W_2^2(\mu_-, \nu_{2-}) \right). \tag{10}$$

It is obvious that

$$W_2^2(\mu_+, \nu_{1+}) = \frac{1}{2} \|x - y_1\|^2, \quad W_2^2(\mu_-, \nu_{2-}) = \frac{1}{2} \|x - y_2\|^2.$$

For $W_2^2(\mu_+, \nu_{1-})$, consider the map

$$T_\#(x + t) \ = \ y_1 - t, \quad t \sim \mathcal{N}(0, \sigma^2 I)$$

we have

$$\begin{aligned} W_2^2(\mu_+, \nu_{1-}) &\leq \mathbb{E}_{u \sim \mu_+} \|u - T_\# u\|^2 \\ &= \mathbb{E}_{u \sim \mu_+} \|u - (y_1 - u + x)\|^2 \\ &= \frac{1}{2} \|x - y_1\|^2 - 4 \mathbb{E}_{u \sim \mu_+} \langle y_1 - x, u - x \rangle + 4 \mathbb{E}_{u \sim \mu_+} \|u - x\|^2 \\ &= \frac{1}{2} \|x - y_1\|^2 - 4 c_1 \sigma \langle \mathbf{m}, y_1 - x \rangle + 4 c_2 \sigma^2, \end{aligned}$$

where c_1 and c_2 are absolute positive constants. Similarly,

$$W_2^2(\mu_-, \nu_{2+}) \leq \frac{1}{2} \|x - y_2\|^2 - 4 c_1 \sigma \langle \mathbf{m}, x - y_2 \rangle + 4 c_2 \sigma^2.$$

Plugging into (10) we get

$$W_2^2(\mu_0 * \mathcal{N}_\sigma, \nu_0 * \mathcal{N}_\sigma) \leq W_2^2(\mu_0, \nu_0) - 4 c_1 \sigma \langle \mathbf{m}, y_1 - y_2 \rangle + 8 c_2 \sigma^2,$$

hence $W_2^2(\mu_0, \nu_0) - W_2^2(\mu_0 * \mathcal{N}_\sigma, \nu_0 * \mathcal{N}_\sigma) \gtrsim \sigma$ for small σ, since $\langle \mathbf{m}, y_1 - y_2 \rangle > 0$.

Finally, we consider the special case where x, y_1, y_2 are on the same line and y_1 is between x and y_2. We choose $\mathbf{m}$ the unit vector along the direction $x - y_1$, and the same line of proof yields the conclusion. □

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