Network change point localisation under local differential privacy (Supplementary Material)

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The supplementary material contains simulation results and all proofs of the main results in Section 3 and Section 4.

A   Numeric results

We generate a sequence of T independent IBNs (Defintion 1) or bipartite IBNs with independent edges (Definition 2) when considering node LDP, with the network size $n_1 = n_2 = n = 50$ and entrywise sparsity level $\rho = 0.4$. There is one and only one change point with a balanced spacing, i.e. the change point $\eta = \Delta = T/2$, where Δ is the minimal spacing. The expectations of the adjacency matrix before and after change point are $\Theta_{\text{pre}} = 0.1\mathbf{1}_{n \times n}$ and $\Theta_{\text{post}} = 0.4\mathbf{1}_{n \times n}$, respectively, where $\mathbf{1}_{n \times n} \in \mathbb{R}^{n \times n}$ has all entries being one. The normalised jump size is therefore $\kappa_0 = \|\Theta_{\text{post}} - \Theta_{\text{pre}}\|_{\text{F}}/(n\rho) = 0.75$. We consider different the minimal spacing Δ and privacy budget α in the simulations.

We use a simplified version of NBS algorithm (Algorithm 1) based on the binary segmentation procedure [e.g. 7]. For small number of change points, our theory still holds for this computationally less demanding algorithm. The thresholding tuning parameter, above which change points are declared, is fixed to be $n \log^{1.5}(T)/10$, $n \log^{1.5}(T)/30$ and $n^2 \log^2(n^2 T)/10$ in the no privacy, edge LDP and node LDP cases, respectively.

Let the estimated set of change points be $\{\hat{\eta}_i\}_{i=1}^{\hat{K}}$ and the true change points be η. We use $\max_i |\hat{\eta}_i - \eta|/\Delta \in [0, 1]$ to evaluate the performances. If no change point is returned, we output one. This is the same as using the scaled two sided Hausdorff distance $d_H(S_1, S_2)/\Delta$ as the metric [e.g. 4, 8] and we expect it to diminish as Δ grows. For any subset $S_1, S_2 \subset \mathbb{Z}$, the Hausdorff distance $d_H(S_1, S_2)$ between S_1 and S_2 is defined as

$$d_H(S_1, S_2) = \max \left\{ \max_{s_1 \in S_1} \min_{s_2 \in S_2} |s_1 - s_2|, \max_{s_2 \in S_2} \min_{s_1 \in S_1} |s_1 - s_2| \right\}.$$

The sets S_1 and S_2 correspond to the set of true change points and estimated change points. If one of S_1 and S_2 is $\emptyset$, then we use the convention $d_H(S_1, S_2) = \Delta$.

The result is collected in Figure 1. Without any privacy constraint, i.e. using raw data, the change can be easily detected with Δ as small as 7. Imposing privacy guarantee requires a larger Δ to consistently localise the change points. The theoretical cost is quantified by our theory under both edge LDP and node LDP. We can see from the three plots in the first row that for the same sample size, the performance deteriorates as α decreases under edge LDP. The node LDP is a more stringent requirement, compared to the edge LDP. From the three plots on the second row, we can see that,

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with the same sample size, the change can be perfectly localised with no error in the no privacy case, and very well localised under edge LDP with $\alpha = 0.1$, but in order to obtain a reasonable estimator, the node information can only be protected at level $\alpha = 1$.
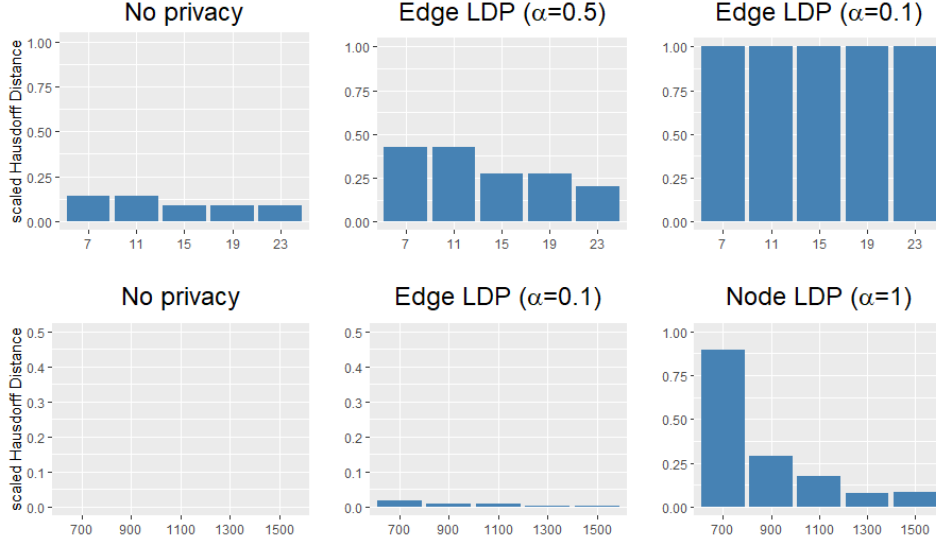
Figure 1: Simulation results. The median of the scaled Hausdorff distance $d_H(S_1, S_2)/\Delta$ over 100 repetitions are plotted against varying minimal spacing Δ on the x-axis, under different privacy constraints. This setting has $\kappa_0 = 0.75$, $\rho = 0.4$, $n_1 = n_2 = n = 50$

B Proofs of results in Section 3

Proof of Lemma 1. Let $\kappa^2 = \frac{n}{68(e^\alpha-1)^2\Delta}$, $v \in \{1,-1\}^n$ and P_v^T be the joint distribution of a collection of independent adjacency matrices $\{A(t)\}_{t=1}^T$ such that

$$\mathbb{E}[A_{ij}(t)] = \rho/2 + \frac{\kappa}{n}(vv^\top)_{ij} \quad 1 \le i \le j \le n, \qquad t \in \{1, \ldots, \Delta\}$$

and

$$\mathbb{E}[A_{ij}(t)] = \rho/2 \quad 1 \le i \le j \le n, \qquad t \in \{\Delta+1, \ldots, T\}.$$

The distribution of each network at time t is denoted by $P_{v,t} = \prod_{1 \le i \le j \le n} P_{v,t,(i,j)}$. Note that $\eta(P_v^T) = \Delta$, $\|A(\Delta) - A(\Delta+1)\|_F^2 = \kappa^2$, and $\kappa_0^2 = \frac{1}{68n\rho^2\Delta(e^\alpha-1)^2}$. We are constrained by $\kappa/n < \rho/2$, which is equivalent to $\kappa_0^2 \le 1/4$. Therefore, for each v, we have $P_v^T \in \mathcal{P}$. Similarly, let $\tilde{P}_v^T$ be the joint distribution of a collection of independent adjacency matrices $\{A(t)\}_{t=1}^T$ such that

$$\mathbb{E}[A_{ij}(t)] = \rho/2 + \frac{\kappa}{n}(vv^\top)_{ij} \quad 1 \le i \le j \le n, \qquad t \in \{T-\Delta+1, \ldots, T\}$$

and

$$\mathbb{E}[A_{ij}(t)] = \rho/2 \quad 1 \le i \le j \le n, \qquad t \in \{1, \ldots, T-\Delta\}.$$

The distribution of each network at time t is denoted by $\tilde{P}_{v,t} = \prod_{1 \le i \le j \le n} \tilde{P}_{v,t,(i,j)}$. Note that $\eta(\tilde{P}_v^T) = T - \Delta$ and $\tilde{P}_v^T \in \mathcal{P}$ for each v. Also, $|\eta(P_v^T) - \eta(\tilde{P}_v^T)| = T - 2\Delta$ for each v. Further, let Z_v^T and $\tilde{Z}_v^T$ be the corresponding joint private distribution generated via some non interactive *edge* α LDP mechanism $Q = \prod_{t=1}^T \prod_{1 \le i \le j \le n} Q_{ijt}(\cdot|A_{ij}(t))$, i.e. for any $1 \le i \le j \le n$

$$Z_v^T = \int Q(\cdot|(A(1))_{ij}, \ldots, A_{ij}(t)) dP_v^T((A(1))_{ij}, \ldots, A_{ij}(t))$$

$$= \int \prod_{t=1}^T \prod_{1 \le i \le j \le n} Q_{ijt}(\cdot|A_{ij}(t)) dP_{v,t,(i,j)}((A(t)_{ij}))$$

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$$= \prod_{t=1}^{T} \prod_{1 \leq i \leq j \leq n} \int Q_{ijt}(\cdot | A_{ij}(t)) dP_{v,t,(i,j)}((A(t)_{ij}))$$

$$= \prod_{t=1}^{T} \prod_{1 \leq i \leq j \leq n} Z_v^{ijt}, \tag{1}$$

and $\tilde{Z}_v^T$ takes a similar form. Write $Z^T = \frac{1}{2^n} \sum_{v \in \{1,-1\}^n} Z_v^T$ and $\tilde{Z}^T = \frac{1}{2^n} \sum_{v \in \{1,-1\}^n} \tilde{Z}_v^T$. It follows from Le Cam's Lemma [e.g. 11] that for $\Delta \leq T/3$

$$\mathcal{R}_{n,\alpha}(\eta(\mathcal{P})) \geq \inf_{Q \in \mathcal{Q}_\alpha^{\text{edge}}} \frac{T}{6}(1 - \text{TV}(\tilde{Z}^T, Z^T)) \geq \inf_{Q \in \mathcal{Q}_\alpha^{\text{edge}}} \frac{\Delta}{6}(1 - \text{TV}(\tilde{Z}^T, Z^T)).$$

To simplify the problem, we write P_0^T as the joint distribution of independent and identically distributed adjacency matrices $\{B(t)\}_{t=1}^T$ such that $\mathbb{E}[B_{ij}(t)] = \rho/2$ for $1 \leq i \leq j \leq n$. The corresponding marginal distribution of the privatised data is denoted by Z_0^T. Now, notice that by triangle inequality and the symmetry of our construction, we have

$$\text{TV}(\tilde{Z}^T, Z^T) \leq 2\text{TV}(Z_0^T, Z^T) \leq \sqrt{2\chi^2(Z^T, Z_0^T)},$$

where the last inequality is due to [5, eq.(2.27)]. In the rest of the proof, we will show that with our choice $\kappa^2 \Delta(e^\alpha - 1)^2 = n/68$, we have $\chi^2(Z^T, Z_0^T) \leq 1/8$ and therefore $\mathcal{R}_{n,\alpha}(\eta(\mathcal{P})) \geq \Delta/12$ as claimed.

To that end, we compute

$$\chi^2(Z^T, Z_0^T) + 1 = \frac{1}{4^n} \sum_{u,v \in \{-1,1\}^n} \mathbb{E}_{Z_0^T}\left(\frac{m(Z_v^T)m(Z_u^T)}{m(Z_0^T)m(Z_0^T)}\right)$$

$$= \frac{1}{4^n} \sum_{u,v \in \{-1,1\}^n} \left[\prod_{t=1}^{T} \prod_{1 \leq i \leq j \leq n} \mathbb{E}_{Z_0^{ijt}}\left(\frac{m(Z_v^{ijt})m(Z_u^{ijt})}{m(Z_0^{ijt})m(Z_0^{ijt})}\right)\right] \tag{2}$$

$$= \frac{1}{4^n} \sum_{u,v \in \{-1,1\}^n} \left[\prod_{t=1}^{\Delta} \prod_{1 \leq i \leq j \leq n} \mathbb{E}_{Z_0^{ijt}}\left(\frac{m(Z_v^{ijt})m(Z_u^{ijt})}{m(Z_0^{ijt})m(Z_0^{ijt})}\right)\right], \tag{3}$$

where Z_0^{ijt} is the distribution of the privatised version of $B_{ij}(t)$ similar to (1) and we use $m(\cdot)$ to denote the density of the corresponding distributions. The last equality is due to $Z_0^{ijt} = Z_u^{ijt} = Z_v^{ijt}$ when $t = \Delta + 1, \ldots, T$.

Write $\Gamma = (\kappa/n)uu^\top$ and $\Lambda = (\kappa/n)vv^\top$, and to simplify notation we use a generic z to denote the privatised data $(z(t))_{ij}$ and write $q_1(z) = q_{ijt}(z|A_{ij}(t) = 1)$ and $q_0(z) = q_{ijt}(z|A_{ij}(t) = 0)$. We further have

$$\mathbb{E}_{Z_0^{ijt}}\left(\frac{m(Z_v^{ijt})m(Z_u^{ijt})}{m(Z_0^{ijt})m(Z_0^{ijt})}\right) \tag{4}$$

$$= \int \frac{\left[q_1(z)(\rho/2 + \Gamma_{ij}) + q_0(z)(1 - \rho/2 - \Gamma_{ij})\right]\left[q_1(z)(\rho/2 + \Lambda_{ij}) + q_0(z)(1 - \rho/2 - \Lambda_{ij})\right]}{\rho q_1(z) + q_0(z)(1 - \rho)} dz$$

$$= \int \frac{\left[(q_1(z) - q_0(z))\rho/2 + q_0(z) + \Gamma_{ij}(q_1(z) - q_0(z))\right]\left[(q_1(z) - q_0(z))\rho/2 + q_0(z) + \Lambda_{ij}(q_1(z) - q_0(z))\right]}{(q_1(z) - q_0(z))\rho/2 + q_0(z)} dz$$

$$= (I) + (II) + (III) \tag{5}$$

where

$$(I) = \int [(q_1(z) - q_0(z))\rho/2 + q_0(z)]dz = 1$$

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$$(II) = \int (q_1(z) - q_0(z))(\Gamma_{ij} + \Lambda_{ij})dz = 0$$

since $q_1(z)$ and $q_0(z)$ are densities of regular conditional probability distributions. Also,

$$(III) = \Gamma_{ij}\Lambda_{ij} \int \frac{(q_1(z) - q_0(z))^2}{(q_1(z) - q_0(z))\rho + q_0(z)}dz = \Gamma_{ij}\Lambda_{ij} \int \frac{(q_0(z))^2(q_1(z)/q_0(z) - 1)^2}{(q_1(z) - q_0(z))\rho + q_0(z)}dz$$
$$= \Gamma_{ij}\Lambda_{ij}C_\alpha$$

with $0 \le C_\alpha \le 2(e^\alpha - 1)^2$ where the last equality is due to Lemma B.1 and $q_1(x)/q_0(x) \in [e^{-\alpha}, e^\alpha]$. Therefore, continue from (4) to see that

$$\mathbb{E}_{Z_0^{ijt}}\left(\frac{m(Z_v^{ijt})m(Z_u^{ijt})}{m(Z_0^{ijt})m(Z_0^{ijt})}\right) \le 1 + 2\Gamma_{ij}\Lambda_{ij}(e^\alpha - 1)^2 \le \exp(2\Gamma_{ij}\Lambda_{ij}(e^\alpha - 1)^2).$$

Continue from (2), with $U, V \in \mathbb{R}^n$ being two independent random vectors with entries being independent Rademacher random variables and $\mathbf{1} \in \mathbb{R}^n$ being a vector of 1's to see that

$$\chi^2(Z^T, Z_0^T) + 1 \le \frac{1}{4^n} \sum_{u,v \in \{-1,1\}^n} \left[\prod_{t=1}^{\Delta} \prod_{1 \le i \le j \le n} \exp(2\Gamma_{ij}\Lambda_{ij}(e^\alpha - 1)^2) \right]$$
$$= \mathbb{E}_{U,V}\left[\exp\left(\frac{2\Delta\kappa^2(e^\alpha - 1)^2}{n^2}(U^\top V)^2 \right) \right]$$
$$= \mathbb{E}_V\left[\exp\left(\frac{2\Delta\kappa^2(e^\alpha - 1)^2}{n^2}(\mathbf{1}^\top V)^2 \right) \right]$$

Let $\epsilon_n = (\mathbf{1}^\top V)^2/n^2$, then

$$\mathbb{E}_V\left[\exp\left(2\Delta\kappa^2(e^\alpha - 1)^2\epsilon_n \right) \right] = \int_0^\infty \mathbb{P}\left(\exp(2\Delta\kappa^2(e^\alpha - 1)^2\epsilon_n) \ge u \right)du$$
$$\le 1 + \int_1^\infty \mathbb{P}\left(\epsilon_n \ge \frac{\log(u)}{2\kappa^2\Delta(e^\alpha - 1)^2} \right)du$$
$$\le 1 + \int_1^\infty 2\exp\left(-\log(u)\frac{n}{4\kappa^2\Delta(e^\alpha - 1)^2} \right)du$$
$$\le 1 + \frac{2}{\frac{n}{4\kappa^2\Delta(e^\alpha - 1)^2} - 1},$$

where the second inequality is Hoeffding's inequality [6, Theorem 2.2.6] and the last inequality holds if

$$\frac{n}{4\kappa^2\Delta(e^\alpha - 1)^2} > 1$$

For $\chi^2(Z^T, Z_0^T) \le 1/8$, it is sufficient to take

$$\frac{n}{\kappa^2\Delta(e^\alpha - 1)^2} \ge 68,$$

which completes the proof. $\qquad\square$

Lemma B.1. *When $\alpha \le 1$ and $\alpha\rho \le 1/2$, then*

$$(q_1(z) - q_0(z))\rho/2 + q_0(z) \ge q_0(z)/2$$

Proof of Lemma B.1. Using the facts $q_1(z)/q_0(z) \ge e^{-\alpha}$, and $e^\alpha - 1 \ge 1 - e^{-\alpha}$, we obtain

$$(q_1(z) - q_0(z))\rho/2 + q_0(z) \ge q_0(z)\left(\frac{q_1(z)}{q_0(z)} - 1\right)\rho/2 + q_0(z) \ge q_0(z)(1 - \rho(1 - e^{-\alpha})/2)$$
$$\ge q_0(z)(1 - \rho(e^\alpha - 1)/2) \ge q_0(z)(1 - \alpha\rho) \ge q_0(z)/2,$$

where in the last two inequalities we use $\alpha \le 1$ and $\alpha\rho \le 1/2$ respectively. $\qquad\square$

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Proof of Lemma 2. The proof parallels the structure of the proof of Lemma 1, so we are somewhat more terse. Let $\kappa^2 = \frac{\sqrt{n_1 n_2}}{20(e^\alpha - 1)^2 \Delta}$, $v \in \{1, -1\}^{n_1}$, P_v^T be the joint distribution of a collection of independent adjacency matrices $\{A(t)\}_{t=1}^T$ such that for $t \in \{1, \ldots, \Delta\}$,

$$\mathbb{E}[A_{i1}(t)] = \rho/2 + \frac{\kappa}{\sqrt{n_1 n_2}} v_i \quad 1 \le i \le n_1, \quad \text{and} \quad A_{i1}(t) = \ldots = A_{in_2}(t),$$

and for $t \in \{\Delta + 1, \ldots, T\}$,

$$\mathbb{E}[A_{i1}(t)] = \rho/2 \quad \text{and} \quad A_{i1}(t) = \ldots = A_{in_2}(t).$$

In words, within each network, the entries of each row are *identical*. In particular, we have for any $1 \le i \le n_1, 1 \le j \le n_2$,

$$\mathbb{E}[A_{ij}(t)] = \rho/2 + \frac{\kappa}{\sqrt{n_1 n_2}} v_i \quad 1 \le t \le \Delta, \qquad \mathbb{E}[A_{ij}(t)] = \rho/2 \quad \Delta + 1 \le t \le T$$

and

$$\mathbb{P}(A_i(t) = \mathbf{1}) = 1 - \mathbb{P}(A_i(t) = \mathbf{0}) = \mathbb{P}(A_{ij}(t) = 1) = \rho/2 + \frac{\kappa}{\sqrt{n_1 n_2}} v_i, \quad 1 \le t \le \Delta$$

$$\mathbb{P}(A_i(t) = \mathbf{1}) = 1 - \mathbb{P}(A_i(t) = \mathbf{0}) = \mathbb{P}(A_{ij}(t) = 1) = \rho/2, \quad \Delta + 1 \le t \le T,$$

where $A_i(t)$ denotes the i-th row of the matrix $A(t)$, and $\mathbf{1} \in \mathbb{R}^{n_2}$, $\mathbf{0} \in \mathbb{R}^{n_2}$ denote a vector of 1's and 0's respectively. The distribution of each network at time t is denoted as $P_{v,t} = \prod_{1 \le i \le n_1} P_v^{it}$.

Note that $\eta(P_v^T) = \Delta$, $\|A(\Delta) - A(\Delta + 1)\|_F^2 = \kappa^2$, and $\kappa_0^2 = \frac{1}{20\sqrt{n_1}\rho^2 \Delta(e^\alpha - 1)^2}$. We are constrained by $\kappa/\sqrt{n_1 n_2} < \rho/2$, which is equivalent to $\kappa_0^2 \le 1/4$. Therefore, for each v, we have $P_v^T \in \mathcal{P}$. Similar to the construction in Lemma 1, we let $\tilde{P}_v^T$ be the joint distribution of a collection of independent adjacency matrices $\{A(t)\}_{t=1}^T$ that is symmetric to P_v^T with respect to time point $T/2$ and has $\eta(\tilde{P}_v^T) = T - \Delta$.

Let Z_v^T and $\tilde{Z}_v^T$ be the corresponding joint private distribution generated via some non interactive *node* α LDP mechanism $Q = \prod_{t=1}^T \prod_{i=1}^{n_1} Q_{it}(\cdot | A_i(t))$, i.e.

$$Z_v^T = \int \prod_{t=1}^T \prod_{1 \le i \le n_1} Q_{it}(\cdot | A_i(t)) dP_v^{it}(A_i(t))$$

$$= \prod_{t=1}^T \prod_{1 \le i \le n_1} Z_v^{it}, \tag{6}$$

and $\tilde{Z}_v^T$ takes a similar form. Write $Z^T = \frac{1}{2^{n_1}} \sum_{v \in \{1, -1\}^{n_1}} Z_v$ and $\tilde{Z}^T = \frac{1}{2^{n_1}} \sum_{v \in \{1, -1\}^{n_1}} \tilde{Z}_v^T$.

Using the same argument as in the proof of Lemma 1, it is sufficient to consider

$$\chi^2(Z^T, Z_0^T) + 1 = \frac{1}{4^{n_1}} \sum_{u, v \in \{-1, 1\}^{n_1}} \left[\prod_{t=1}^\Delta \prod_{1 \le i \le n_1} \mathbb{E}_{Z_0^{it}} \left(\frac{m(Z_v^{it}) m(Z_u^{it})}{m(Z_0^{it}) m(Z_0^{it})} \right) \right],$$

where $Z_0^{it} = \int Q_{it}(\cdot | (A(t))_i) dP_v^{iT}((A(t))_i)$ and $m(\cdot)$ denotes the density of the corresponding distributions, and show that $\chi^2(Z^T, Z_0^T) \le 1/8$ with our choice $\kappa^2 = \frac{\sqrt{n_1 n_2}}{20(e^\alpha - 1)^2 \Delta}$.

We use a generic z to denote the i-th row of the private network $z(t)$ and write $q_1(z) = q_{it}(z | A_i(t) = \mathbf{1})$ and $q_0(z) = q_{it}(z | A_i(t) = \mathbf{0})$. Following the same calculation as in (4), we have

$$\mathbb{E}_{Z_0^{it}} \left(\frac{m(Z_v^{it}) m(Z_u^{it})}{m(Z_0^{it}) m(Z_0^{it})} \right) = 1 + \frac{\kappa^2}{n_1 n_2} u_i v_i \int \frac{(q_0(z))^2 (q_1(z)/q_0(z) - 1)^2}{(q_1(z) - q_0(z))\rho + q_0(z)} dz$$

$$\le 1 + \frac{2\kappa^2}{n_1 n_2} u_i v_i \int q_0(z)(q_1(z)/q_0(z) - 1)^2 dz$$

$$\le 1 + \frac{2\kappa^2}{n_1 n_2} u_i v_i (e^\alpha - 1)^2$$

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$$\leq \exp\Big(\frac{2\kappa^2}{n_1 n_2} u_i v_i (e^\alpha - 1)^2\Big)$$

where the first inequality is due to Lemma B.1. Next, writing $U \in \mathbb{R}^{n_1}$ as a random vector with independent Rademacher entries and $\epsilon = \mathbf{1}^\top U / n_1$, we have

$$\chi^2(Z^T, Z_0^T) + 1$$

$$\leq \mathbb{E}_U \Bigg[\exp\Big(\frac{2\Delta\kappa^2(e^\alpha - 1)^2}{n_1 n_2}(\mathbf{1}^\top U)\Big) \Bigg]$$

$$= \int_0^\infty \mathbb{P}\Big(\exp(2\Delta\kappa^2(e^\alpha - 1)^2 n_2^{-1}\epsilon) \geq u\Big) du$$

$$\leq 1 + \int_1^e \mathbb{P}\Bigg(\epsilon \geq \frac{n_2 \log(u)}{2\kappa^2\Delta(e^\alpha - 1)^2}\Bigg) du + \int_e^\infty \mathbb{P}\Bigg(\epsilon \geq \frac{n_2 \log(u)}{2\kappa^2\Delta(e^\alpha - 1)^2}\Bigg) du$$

$$\leq 1 + \int_1^e \exp\Big(-(\log(u))^2 \frac{n_1 n_2^2}{(2\kappa^2\Delta(e^\alpha - 1)^2)^2}\Big) + \int_e^\infty \exp\Big(-\log(u)\frac{n_1 n_2^2}{(2\kappa^2\Delta(e^\alpha - 1)^2)^2}\Big) du,$$

where the last inequality is Hoeffding's inequality. Writing $x = \frac{n_1 n_2^2}{(2\kappa^2\Delta(e^\alpha - 1)^2)^2}$, we have for any $x > 1$

$$\chi^2(Z^T, Z_0^T) \leq \int_1^e \exp(-x \log^2(u)) du - \frac{1}{1-x}$$

With the choice $x \geq 90$, it holds that

$$\chi^2(Z^T, Z_0^T) \leq 0.1 + 0.012 \leq 0.125.$$

Therefore, it is sufficient to take

$$\frac{\sqrt{n_1} n_2}{\kappa^2 \Delta(e^\alpha - 1)^2} = 20.$$

to ensure $\chi^2(Z^T, Z_0^T) \leq 1/8$, which completes the proof.

$\square$

C   Proof of results in Section 4

Proof of Theorem 3. We write $q = (1 + e^\alpha)^{-1}$, the corruption probability that $A'_{ij}(t) \neq A_{ij}(t)$ in (8). The proof relies on the observation that if $X \sim \text{Bernoulli}(\theta)$ then the privatised Z obtained by (8) is distributed as $\text{Bernoulli}(q * \theta)$ where $q * \theta := q(1-\theta) + (1-q)\theta = q + (1-2q)\theta$. This implies if X is the adjacency matrix of an inhomogeneous Bernoulli network and Z is a corresponding private view generated by (8) with corruption probability q, then Z is distributed as an inhomogeneous Bernoulli model with parameter matrix $q * \Theta$, where $(q * \Theta)_{ij} = q * \theta_{ij}$. In addition, the change point structure is preserved after the privatisation but with

$$\min_{k=1,\dots,K+1} \|q * \Theta(\eta_k) - q * \Theta(\eta_k - 1)\|_{\mathrm{F}} = (1-2q)\|\Theta(\eta_k) - \Theta(\eta_k - 1)\|_{\mathrm{F}}.$$

Also, since $q * \theta$ is monotonic increasing in θ (for $q < 1/2$), we have $\rho' := \|q * \Theta(t)\|_\infty = q + (1-2q)\rho$ for any $t = 1, \dots, T$. Lastly, we have $\rho' \geq q \geq (1 + e)^{-1}$, where the second inequality holds when $\alpha \leq 1$, and $(1 + e)^{-1} \geq \log(n)/n$ for any $n > 1$, which guarantees the sparsity assumption in [9] is satisfied for the privatised inhomogeneous Bernoulli network $(A'(1), \dots, A'(T))$.

The result now follows by a direct application of Theorem 1 in [9] but with some different model parameters representing the effects of privatisation, i.e. $(K' = K, \Delta' = \Delta, \kappa'_0, n' = n, \rho')$, where $\kappa'_0 = \frac{(1-2q)\rho}{\rho'}\kappa_0$. Using the transformed parameters, the Assumption 2 in [9] becomes

$$(1-2q)\kappa_0 \sqrt{\frac{\rho}{1 - 2q + q/\rho}} \geq C_\alpha \sqrt{\frac{1}{n\Delta}} \log^{1+\xi}(T),$$

and the localisation rate in Theorem 1 in [9] becomes

$$\epsilon = C_1 \log(T)\left(\frac{\sqrt{\Delta}}{(1-2q)\kappa_0 n\rho} + \frac{(1 - 2q + q/\rho)\sqrt{\log(T)}}{(1-2q)^2\kappa_0^2 n\rho}\right).$$

Substituting $q = (1 + e^\alpha)^{-1}$ yields the claimed result (with different constants). $\square$

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Algorithm 1 Privacy mechanism for ℓ_∞-ball with radius 1 [1, 2]

INPUT: a vector $V \in \mathbb{R}^d$ with $\|V\|_\infty \leq 1$, privacy parameter α

1. Generate $\widetilde{V}$ with independent coordinates according to

$$\mathbb{P}(\tilde{V}_j = 1|V_j) = 1 - \mathbb{P}(\tilde{V}_j = -1|V_j) = \frac{1}{2} + \frac{V_j}{2}.$$

2. Let $T \sim \mathrm{Ber}(e^\alpha/(e^\alpha + 1))$ be independent of $\widetilde{V}$. Generate Z according to

$$Z \sim \begin{cases} \mathrm{Uniform}\left(z \in \{-B, B\}^d \middle| \langle z, \tilde{V} \rangle \geq 0\right), & T = 1, \\ \mathrm{Uniform}\left(z \in \{-B, B\}^d \middle| \langle z, \tilde{V} \rangle \leq 0\right), & T = 0, \end{cases}$$

where

$$B = C_d \frac{e^\alpha + 1}{e^\alpha - 1} \quad \text{and} \quad C_d^{-1} = \begin{cases} \frac{1}{2^{d-1}}\binom{d-1}{(d-1)/2}, & d \quad \text{odd}, \\ \frac{1}{2^{d-1}+\frac{1}{2}\binom{d}{d/2}}\binom{d-1}{d/2}, & d \quad \text{even}. \end{cases}$$

OUTPUT: The privatised vector Z.

Proof of Lemma 5. We first restate the algorithm in Algorithm 1. For simplicity, we write Σ for Σ_Z. We start with proving (16). Given (14) and (15), we have when d is odd,

$$\|\Sigma\| = \|B^2 I - \mathbb{E}(V)(\mathbb{E}(V))^\top\| \leq B^2 + \|\mathbb{E}(V)\|_2^2,$$

and similarly when d is even, we have

$$\|\Sigma\| \leq B^2 + \|\mathbb{E}(V)\|_2^2 + \frac{C_1\sqrt{d}}{\alpha^2}\sqrt{\max_{i,j}\mathbb{E}(V_iV_j)}.$$

To see (14), simply note that

$$\mathrm{Var}(Z_k) = \mathbb{E}Z_k^2 - (\mathbb{E}Z_k)^2 = B^2 - (\mathbb{E}V_k)^2$$

since $\mathbb{E}Z_k = \mathbb{E}V_k$ for any $k = 1, \ldots, d$.

To prove (15), we start with for any $i, j = 1, \ldots, d$ with $i \neq j$

$$\mathrm{Cov}(Z_i, Z_j) = \mathbb{E}[\mathrm{Cov}(Z_i, Z_j|V)] + \mathrm{Cov}(\mathbb{E}(Z_i|V), \mathbb{E}(Z_j|V)) = \mathbb{E}[\mathrm{Cov}(Z_i, Z_j|V)] + \mathrm{Cov}(V_i, V_j),$$
$$(7)$$

where we use the unbiased property $\mathbb{E}(Z_i|V) = V_i$ for any $i = 1, \ldots, d$ [see Appendix I.3 in 2]. Note that

$$\mathrm{Cov}(Z_i, Z_j|V = v) = \mathbb{E}(Z_iZ_j|V = v) - v_iv_j = \sum_{\tilde{v}\in\{-1,1\}^d}\mathbb{E}(Z_iZ_j|\tilde{v})\mathbb{P}(\tilde{v}|v) - v_iv_j \quad (8)$$

and

$$\mathbb{E}(Z_iZ_j|\tilde{v}) = \pi_\alpha\mathbb{E}(Z_iZ_j|\langle z, \tilde{v} \rangle \geq 0) + (1 - \pi_\alpha)\mathbb{E}(Z_iZ_j|\langle z, \tilde{v} \rangle \leq 0). \quad (9)$$

Therefore, we need to compute $\mathbb{E}(Z_iZ_j|\langle z, \tilde{v} \rangle \geq 0)$ and $\mathbb{E}(Z_iZ_j|\langle z, \tilde{v} \rangle \leq 0)$. We consider the cases of d being odd and even separately below.

When d is odd: we have

$$\sum_{z:\langle z,\tilde{v}\rangle\geq 0} z_iz_j = \sum_{l=0}^{(d-1)/2} B^2\tilde{v}_i\tilde{v}_j\left(\binom{d-2}{l} + \binom{d-2}{l-2} - 2\binom{d-2}{l-1}\right)$$

$$= B^2\tilde{v}_i\tilde{v}_j\left(\sum_{l=0}^{(d-1)/2}\binom{d-2}{l} + \sum_{l=0}^{(d-5)/2}\binom{d-2}{l} - 2\sum_{l=0}^{(d-3)/2}\binom{d-2}{l}\right)$$

$$= B^2\tilde{v}_i\tilde{v}_j\left(\binom{d-2}{(d-1)/2} - \binom{d-2}{(d-3)/2}\right)$$

$$= 0$$

Therefore $\mathbb{E}(Z_iZ_j|\langle z, \tilde{v} \rangle \geq 0) = 0$ and by symmetry $\mathbb{E}(Z_iZ_j|\langle z, \tilde{v} \rangle \leq 0) = 0$. Hence, $\mathrm{Cov}(Z_i, Z_j) = -\mathbb{E}(V_iV_j) + \mathrm{Cov}(V_i, V_j) = -\mathbb{E}(V_i)\mathbb{E}(V_j)$ when d is odd.

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When d is even: We have

$$\sum_{z:\langle z,\tilde{v}\rangle \geq 0} z_i z_j = \sum_{l=0}^{d/2} B^2 \tilde{v}_i \tilde{v}_j \left(\binom{d-2}{l} + \binom{d-2}{l-2} - 2\binom{d-2}{l-1} \right)$$

$$= B^2 \tilde{v}_i \tilde{v}_j \left(\sum_{l=0}^{d/2} \binom{d-2}{l} + \sum_{l=0}^{d/2-2} \binom{d-2}{l} - 2\sum_{l=0}^{d/2-1} \binom{d-2}{l} \right)$$

$$= B^2 \tilde{v}_i \tilde{v}_j \left(\binom{d-2}{d/2} - \binom{d-2}{d/2-1} \right)$$

$$= -\frac{2B^2}{d} \tilde{v}_i \tilde{v}_j \binom{d-2}{d/2-1}$$

where the last equality is due to $k\binom{n}{k} = (n-k+1)\binom{n}{k-1}$. Since the set $\{z \in \{-1,1\}^d | \langle z, \tilde{v}\rangle \geq 0\}$ has cardinality $M = 2^{d-1} + \frac{1}{2}\binom{d}{d/2}$, we have

$$\mathbb{E}(Z_i Z_j | \langle z, \tilde{v}\rangle \geq 0) = -\frac{2B^2}{dM} \tilde{v}_i \tilde{v}_j \binom{d-2}{d/2-1}.$$

When $\langle z, \tilde{v}\rangle \leq 0$, we obtain the *same* result

$$\sum_{z:\langle z,\tilde{v}\rangle \leq 0} z_i z_j = \sum_{l=d/2}^{d} B^2 \tilde{v}_i \tilde{v}_j \left(\binom{d-2}{l} + \binom{d-2}{l-2} - 2\binom{d-2}{l-1} \right)$$

$$= B^2 \tilde{v}_i \tilde{v}_j \left(\sum_{l=d/2}^{d} \binom{d-2}{l} + \sum_{l=d/2-2}^{d} \binom{d-2}{l} - 2\sum_{l=d/2-1}^{d} \binom{d-2}{l} \right)$$

$$= B^2 \tilde{v}_i \tilde{v}_j \left(\binom{d-2}{d/2} - \binom{d-2}{d/2-1} \right)$$

$$= -\frac{2B^2}{d} \tilde{v}_i \tilde{v}_j \binom{d-2}{d/2-1}.$$

Then, from (9) we get

$$\mathbb{E}(Z_i Z_j | \tilde{v}) = \mathbb{E}(Z_i Z_j | \langle z, \tilde{v}\rangle \geq 0) = -\frac{2B^2}{dM} \tilde{v}_i \tilde{v}_j \binom{d-2}{d/2-1}.$$

Now we look at

$$\frac{2B^2}{dM} \binom{d-2}{d/2-1} = \left(\frac{e^\alpha + 1}{e^\alpha - 1} \right)^2 \left(\frac{2^{d-1} + \frac{1}{2}\binom{d}{d/2}}{\binom{d-1}{d/2}} \right)^2 \frac{\binom{d-2}{d/2-1}}{d(2^{d-1} + \frac{1}{2}\binom{d}{d/2})}$$

$$= \frac{1}{d} \left(\frac{e^\alpha + 1}{e^\alpha - 1} \right)^2 \left(\frac{2^{d-1} + \frac{1}{2}\binom{d}{d/2}}{\frac{d-1}{d/2}\binom{d-2}{d/2-1}} \right)$$

$$= \frac{c_\alpha}{d\alpha^2} \frac{2^{d-1} + c_d 2^{d-1}/\sqrt{d}}{\frac{d-1}{d/2} c_{d-2} 2^{d-2}/\sqrt{d-2}}$$

$$= \frac{C_{d,\alpha}}{\sqrt{d}\alpha^2}$$

where we use Stirling approximation to obtain $\binom{d}{d/2} = c_d 2^d/\sqrt{d}$ with $\exp(-1/6)(\sqrt{2\pi})^{-1} < c_d < (\sqrt{2\pi})^{-1}$ and $1/4 < c_\alpha < (e+1)^2$. Indeed, using the non-asymptotic inequalities for any even $d \geq 2$

$$\sqrt{2\pi d}(d/e)^d \exp\left(\frac{1}{12d+1} \right) < d! < \sqrt{2\pi d}(d/e)^d \exp\left(\frac{1}{12d} \right) \tag{10}$$

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we have

$$\binom{d}{d/2} = \frac{d!}{((d/2)!)^2} \leq \frac{\exp\left(\frac{1}{12d} - \frac{2}{6d+1}\right)}{\sqrt{2\pi}\sqrt{d}2^{-d}} \leq \frac{2^d}{\sqrt{2\pi d}}$$

and similarly

$$\binom{d}{d/2} = \frac{d!}{((d/2)!)^2} \geq \frac{\exp\left(\frac{1}{12d+1} - \frac{1}{3d}\right)}{\sqrt{2\pi}\sqrt{d}2^{-d}} \geq \frac{\exp(-1/6)2^d}{\sqrt{2\pi d}}.$$

Therefore, from (8) we have that there exists $C_0 < C_{d,\alpha} < C_1$ with C_0, C_1 being absolute constants such that

$$\mathrm{Cov}(Z_i, Z_j | V = v) = -\frac{C_{d,\alpha}}{\sqrt{d}\alpha^2}\mathbb{E}(\tilde{V}_i\tilde{V}_j|v) - v_i v_j = -\left(1 + \frac{C_{d,\alpha}}{\sqrt{d}\alpha^2}\right)v_i v_j.$$

It follows from (7) that when d is even

$$\mathrm{Cov}(Z_i, Z_j) = -\left(1 + \frac{C_{d,\alpha}}{\sqrt{d}\alpha^2}\right)\mathbb{E}(V_iV_j) + \mathrm{Cov}(V_i, V_j) = -\frac{C_{d,\alpha}}{\sqrt{d}\alpha^2}\mathbb{E}(V_iV_j) - \mathbb{E}(V_i)\mathbb{E}(V_j).$$

$\square$

Proof of Theorem 4. First, we set

$$\kappa^2 := \min_{k=1,\ldots,K} \|\Theta(\eta_k) - \Theta(\eta_k - 1)\|_F^2 = \kappa_0^2 n_1 n_2 \rho^2,$$

to be the unnormalised minimal jump size in Frobenius norm and

$$\varepsilon = c_5 \log(Tn_1n_2)\left(\frac{\sqrt{\Delta}}{\kappa_0 \rho \alpha}\sqrt{\frac{n_2}{n_1}} + \frac{\log(Tn_1n_2)}{\rho^2\alpha^2\kappa_0^2}\max\left\{\sqrt{\frac{n_2}{n_1}}, \frac{n_2}{n_1}\right\}\right) \tag{11}$$

to be the claimed localisation error in Theorem 4.

In the proof we use the notation κ^2 and it translates directly to the signal to noise condition (12) in terms of κ_0^2. We also use $U(t)$ and $V(t)$ to denote the privatised matrices obtained by (11), which is consistent with the notations in Algorithm 1 and results in Appendix C.1.

We consider two events. The first event guarantees the quality of the randomly generated intervals. Let $\{\alpha_m\}_{m=1}^M$ and $\{\beta_m\}_{m=1}^M$ be two independent sequences selected uniformly randomly from $\{1, \ldots, T\}$.

$$\mathcal{M} = \bigcap_{k=1}^K \{\alpha \in S_k, \beta_m \in E_k, \text{for some } m \in \{1, \ldots, M\}\},$$

where $S_k = [\eta_k - 3\Delta/4, \eta_k - \Delta/2]$ and $E_k = [\eta_k + \Delta/2, \eta_k + 3\Delta/4]$, $k = 1, \ldots, K$. It is shown in Wang et al. [9, Lemma 24] that

$$\mathbb{P}(\mathcal{M}) \geq 1 - \exp\left(\log(T/\Delta) - \frac{M\Delta^2}{16T^2}\right).$$

Next, for $0 \leq s < t < e \leq T$, consider the events

$$\mathcal{A}(s,t,e) = \left\{\left|(\tilde{U}^{(s,e)}(t), \tilde{V}^{(s,e)}(t)) - \|\tilde{\Theta}^{(s,e)}(t)\|_F^2\right|\right.$$

$$\left. \leq C_\beta B \log(Tn_1n_2)\left(\sqrt{n_2}\|\tilde{\Theta}^{(s,e)}(t)\|_F + B\log(Tn_1n_2)\max\{\sqrt{n_1n_2}, n_2\}\right)\right\}.$$

Choosing c, c' and $c'' > 3$ in Lemma C.3, we have

$$\mathbb{P}(\mathcal{A}) = \mathbb{P}\left(\bigcup_{1 \leq s < t < e \leq T} \mathcal{A}(s,t,e)\right) \geq 1 - (T^{3-c_4} + T^{3-c_5} + 2T^{3-c_6})$$

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by a union bound. The rest of the proof is conditional on the event $\mathcal{A} \cap \mathcal{M}$ and does not involve further probabilistic arguments.

Our proof follows the standard induction-like argument for proving consistency of change point estimators [9, 3, 10]. In particular, since the effects of node LDP is fully represented in the probabilistic arguments of analysing event $\mathcal{A}$ (c.f. Lemma C.3 and Lemma 6 in Wang et al. [9]), the rest of the analytic arguments in the proof of Theorem 1 in Wang et al. [9] can be applied to our problem directly. Therefore, we only point out the differences in each step between their proof and ours caused by the different concentration behaviour of $(\tilde{U}^{(s,e)}(t), \tilde{V}^{(s,e)}(t))$. To that end, we consider a generic time interval $(s, e) \subset (0, T)$ that satisfies

$$\eta_{r-1} \le s \le \eta_r \le \ldots \le \eta_{r+1} \le e \le \eta_{r+q+1}, \quad q \ge -1$$

and

$$\max\{\min\{\eta_r - s, s - \eta_{r-1}\}, \min\{\eta_{r+q+1} - e, e - \eta_{r+q}\}\} \le \varepsilon,$$

where $q = -1$ means that there is no change point contained in (s, e) and ε is given in (11). A change point η_p in $[s, e]$ is referred to as undetected if $\min\{\eta_p - s, \eta_p - e\} \ge 3\Delta/4$. Let s_m, e_m, a_m, b_m, τ and m^* be defined as in the algorithm. The next four steps parallel the four steps in the proof of Theorem 1 in [9] and establish that our algorithm

1. rejects the existence of undetected change points if (s, e) does not contain any undetected change points

2. output an estimate b such that $|\eta_p - b| \le \varepsilon$ if these is at least one undetected change point in (s, e).

Step 1. Suppose that there do not exist any undetected change points within (s, e). We have with

$$\tau > c_1 n_2 \alpha^{-2} \log^2(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\},$$

the algorithm will always correctly reject the existence of undetected change points.

Step 2. Suppose that there exists an undetected change point $\eta_p \in (s, e)$. On the event $\mathcal{M}$, there exists an interval $[s_m, e_m]$ such that

$$\eta_p - 3\Delta/4 \le s_m \le \eta_p - \Delta/8 \quad \text{and} \quad \eta_p + \Delta/8 \le e_m \le \eta_p + 3\Delta/4.$$

Now, on event $\mathcal{A}$, we have

$$(\tilde{U}^{(s_m, e_m)}(\eta_p), \tilde{V}^{(s_m, e_m)}(\eta_p)) \ge \|\tilde{\Theta}^{(s_m, e_m)}(\eta_p)\|_{\mathrm{F}}^2$$
$$- C_\beta B \log(T n_1 n_2) \left(\sqrt{n_2} \|\tilde{\Theta}^{(s,e)}(\eta_p)\|_{\mathrm{F}} + B \log(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\} \right)$$

It follows from Wang et al. [9, Lemma 17] that

$$\|\tilde{\Theta}^{(s_m, e_m)}(\eta_p)\|_{\mathrm{F}}^2 \ge \kappa^2 \Delta/8.$$

Then using (12) we have

$$\kappa^2 \Delta/16 \ge \frac{c_0}{16} \frac{n_2}{\alpha^2} \log^{2+\xi}(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\} \ge C_\beta B^2 \log^2(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\}$$
(12)

if $c_0 \log^\xi(T n_1 n_2) > 16 C_\beta$. Also,

$$\frac{\|\tilde{\Theta}^{(s_m, e_m)}(\eta_p)\|_{\mathrm{F}}}{2} \ge \frac{\kappa \sqrt{\Delta}}{2\sqrt{2}} \ge \frac{\sqrt{c_0}}{2\sqrt{2}} B \log^{1+\xi/2}(T n_1 n_2) (\max\{\sqrt{n_1 n_2}, n_2\})^{1/2} \ge$$
$$C_\beta B \sqrt{n_2} \log(T n_1 n_2) \quad (13)$$

provided $c_0 \log^\xi(T n_1 n_2) > 8 C_\beta^2$. Therefore, we have for c_0 large enough, there exists some absolute constant c_2 such that

$$(\tilde{U}^{(s_m, e_m)}(\eta_p), \tilde{V}^{(s_m, e_m)}(\eta_p)) \ge c_2 \kappa^2 \Delta.$$

By the definition of m^*, we have

$$(\tilde{U}^{(s_{m*}, e_{m*})}(b_{m*}), \tilde{V}^{(s_{m*}, e_{m*})}(b_{m*})) \ge c_2 \kappa^2 \Delta.$$
(14)

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Thus, with $\tau < c_2 \kappa^2 \Delta$, our algorithm can consistently detect the existence of undetected change points.

Step 3. Suppose that there exists at least one undetected change point $\eta_p \in (s, e)$. We show that the selected interval (s_{m*}, e_{m*}) indeed contains an undetected change point η_p. Suppose that

$$\max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F^2 < c_2 \kappa^2 \Delta / 2 \tag{15}$$

Then

$$\begin{aligned}
&\max_{s_{m*} < t < e_{m*}} (\tilde{U}^{(s_{m*}, e_{m*})}(t), \tilde{V}^{(s_{m*}, e_{m*})}(t)) \\
&\leq \max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F^2 + C_\beta B \log(T n_1 n_2) \Big(\sqrt{n_2} \max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F \\
&\quad + B \log(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\} \Big) \\
&\leq c_2 \kappa^2 \Delta / 2 + \sqrt{c_2/2} \kappa \sqrt{\Delta} C_\beta B \log(T n_1 n_2) \sqrt{n_2} + C_\beta B^2 \log^2(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\} \\
&\leq c_2 \kappa^2 \Delta
\end{aligned}$$

where the first inequality is due to the definition of event $\mathcal{A}$, the second inequality is due to (15), and the last inequality is due to (12) with sufficiently large c_0. This is a contradiction to (14), and therefore

$$\max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F^2 > c_2 \kappa^2 \Delta / 2 \tag{16}$$

Then, we can conclude that $[s_{m*}, e_{m*}]$ contains at least one undetected change point using the same argument as that in Step 3 in [9].

Step 4. Continue from Step 3, we will show that

$$|b_{m*} - \eta_p| \leq \varepsilon,$$

by applying Lemma 7 in [9]. The conditions of Lemma 7 can be easily checked by letting

$$\begin{aligned}
\lambda &= \max_{s_{m*} < t < e_{m*}} |(\tilde{U}^{(s_{m*}, e_{m*})}(t), \tilde{V}^{(s_{m*}, e_{m*})}(t)) - \|\tilde{\Theta}^{s_{m*}, e_{m*}}(t)\|_F^2| \\
&\leq C_\beta B \log(T n_1 n_2) \Big(\sqrt{n_2} \max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F + B \log(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\} \Big) \\
&\leq c_3 \max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F^2
\end{aligned}$$

where the first inequality is due to the definition of event $\mathcal{A}$ and the second inequality is obtained by combining (16), (12) and (13). Then their Lemma 7 guarantees that there exists an undetected change point η_p within $[s, e]$ with

$$|\eta_p - b| \leq \frac{C_3 \Delta \lambda}{\|\tilde{\Theta}^{s_{m*}, e_{m*}}(\eta_p)\|_F^2} \quad \text{and} \quad \|\tilde{\Theta}^{s_{m*}, e_{m*}}(\eta_p)\|_F^2 \geq c' \max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F^2.$$

Combing with (16), we have

$$\begin{aligned}
&|\eta_p - b| \\
&\leq \frac{C_3 \Delta C_\beta B \log(T n_1 n_2) \Big(\sqrt{n_2} \max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F + B \log(T n_1 n_2) \max\{\sqrt{n_1 n_2}, n_2\} \Big)}{c' \max_{s_{m*} < t < e_{m*}} \|\tilde{\Theta}^{(s_{m*}, e_{m*})}(t)\|_F^2} \\
&= c_7 B \log(T n_1 n_2) \left(\frac{\sqrt{\Delta n_2}}{\kappa} + \frac{B \max\{\sqrt{n_1 n_2}, n_2\} \log(T n_1 n_2)}{\kappa^2} \right) \\
&= c_8 \log(T n_1 n_2) \left(\frac{\sqrt{\Delta} n_2}{\kappa \alpha} + \frac{\max\{\sqrt{n_1 n_2}, n_2\} n_2 \log(T n_1 n_2)}{\alpha^2 \kappa^2} \right) \\
&= \varepsilon,
\end{aligned}$$

which completes the proof. $\qquad\square$

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C.1 Probability bounds

In this section, we derive necessary probability bounds for bipartite node privacy. Recall that X_i denotes the ith *row* of some general matrix X, $X^\top$ denotes the transpose of X, and $\|X\|$ denotes the operator norm of X. In particular we consider two independent copies $\{X(t)\}_{t=1}^T$ and $\{Y(t)\}_{t=1}^T$ satisfying Assumption 1. Let $\{U(t)\}_{t=1}^T$ and $\{V(t)\}_{t=1}^T$ be their private versions obtained by applying the sampling mechanism (11) to $\{X_i(t)\}_{t=1}^T$ and $\{Y_i(t)\}_{t=1}^T$ respectively. Note that

$$\mathbb{E}(U(t)) = \mathbb{E}(V(t)) = \mathbb{E}(X(t)) = \mathbb{E}(Y(t)) = \Theta(t)$$

since $\{U(t)\}_{t=1}^T$ and $\{V(t)\}_{t=1}^T$ are also independent copies and the sampling mechanism is unbiased. We write

$$\tilde{U} = \sum_{t=1}^T w_t U(t), \quad \tilde{V} = \sum_{t=1}^T w_t V(t) \quad \text{and} \quad \tilde{\Theta} = \sum_{t=1}^T w_t \Theta(t)$$

with

$$\sum_{t=1}^T w_t^2 = 1.$$

Also we write $\Sigma_i(t)$ as the covariance matrix for $U_i(t)$ and $V_i(t)$ and applying (16) with $d = n_2$ yields that when $\alpha < 1$ and n_2 is odd,

$$\|\Sigma_i(t)\| \le B^2 + \|\Theta_i(t)\|_2^2 \le B^2 + n_2\rho^2 \le 2B^2,$$

for any $i = 1, \ldots, n_1$ and $t = 1, \ldots, T$. Similarly when $\alpha < 1$ and n_2 is even

$$\|\Sigma_i(t)\| \le B^2 + n_2\rho^2 + \frac{c\sqrt{n_2}}{\alpha^2} \le 3B^2.$$

Therefore, we have

$$\max_{i=1,\ldots,n_1, t=1,\ldots,T} \|\Sigma_i(t)\| \le 3B^2 \tag{17}$$

for both n_2 is odd and even cases.

Lemma C.1. *Let $k_i \in \mathbb{R}^{n_2}$ be an arbitrary vector. Then for any $\varepsilon > 0$, we have*

$$\mathbb{P}\left(\left|\sum_{t=1}^T w_t \sum_{i=1}^{n_1} k_i^\top (V_i(t) - \Theta_i(t))\right| > \epsilon\right)$$

$$\le 2\exp\left(\frac{-\frac{1}{2}\epsilon^2}{3B^2 \sum_{i=1}^{n_1} \|k_i\|_2^2 + \max_{i=1,\ldots,n_1} \|k_i\|_2 \sqrt{n_2} B 2\epsilon/3}\right).$$

Proof. The proof is due to an application of Bernstein's inequality [6, Theorem 2.8.4]. Notice that

$$\mathbb{E}\left(\sum_{t=1}^T w_t \sum_{i=1}^{n_1} k_i^\top (V_i(t) - \Theta_i(t))\right)^2 = \sum_{t=1}^T w_t^2 \sum_{i=1}^{n_1} \mathbb{E}[k_i^\top (V_i(t) - \Theta_i(t))]^2$$

$$= \sum_{t=1}^T w_t^2 \sum_{i=1}^{n_1} \mathbb{E}\left[\sum_{j=1}^{n_2} k_{ij}(V_{ij}(t) - \Theta_{ij}(t))\right]^2$$

$$= \sum_{t=1}^T w_t^2 \sum_{i=1}^{n_1} k_i^\top \Sigma_i(t) k_i$$

$$\le \sum_{i=1}^{n_1} \|k_i\|_2^2 \max_{i,t} \|\Sigma_i(t)\|$$

$$\le 3B^2 \sum_{i=1}^{n_1} \|k_i\|_2^2$$

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where the first line is due to the independence across t and $i = 1, \ldots, n_1$, the first inequality is due to the definition of operator norm $\|\Sigma_i\|$ and $\sum_{t=1}^{T} w_t^2 = 1$, and in the last line we use (17). Also since $|k_i^\top (V_i(t) - \Theta_i(t))| \le \|k_i\|_2 \|V_i(t) - \Theta_i(t)\|_2 \le 2\|k_i\|_2 \sqrt{n_2} B$ and $w_t \le 1$, we have

$$\mathbb{P}\left(\left| \sum_{t=1}^{T} w_t \sum_{i=1}^{n_1} k_i^\top (V_i(t) - \Theta_i(t)) \right| > \epsilon \right) \le 2 \exp\left(\frac{-\frac{1}{2}\epsilon^2}{3B^2 \sum_{i=1}^{n_1} \|k_i\|_2^2 + \max_{i=1,\ldots,n_1} \|k_i\|_2 \sqrt{n_2} B 2\epsilon/3} \right)$$

by Bernstein's inequality, as claimed. $\qquad\square$

Lemma C.2. *Let $k_i = \sum_{t=1}^{T} w_t (V_i(t) - \Theta_i(t))$. Then there exist absolute constants $C, c > 0$ such that*

$$\mathbb{P}\left(\max_{i=1,\ldots,n_1} \|k_i\|_2 \ge C\sqrt{n_2} B \log(Tn_1n_2) \right) \le T^{-c}.$$

Proof. First note that $\|V_i(t) - \Theta_i(t)\|_2 \le 2\sqrt{n_2} B$ for any $i = 1, \ldots, n_1$ and $t = 1, \ldots, T$. Also, we have

$$\mathbb{E}\|V_i(t) - \Theta_i(t)\|_2^2 \le \mathbb{E}\|V_i(t)\|_2^2 = n_2 B^2 \quad \text{and} \quad \mathbb{E}(V_i(t) - \Theta_i(t))(V_i(t) - \Theta_i(t))^\top = \Sigma_i(t),$$

and $\max_{i,t} \|\Sigma_i(t)\| \le 3B^2$. Next, we apply the matrix Bernstein inequality for rectangular matrices [6, Exercise 5.4.15] to $k_i = \sum_{t=1}^{T} w_t (V_i(t) - \Theta_i(t))$, for any fixed $i = 1, \ldots, n_1$, and obtain

$$\mathbb{P}(\|k_i\|_2 \ge t) \le 2(n_2 + 1) \exp\left(-\frac{t^2/2}{\sigma^2 + \sqrt{n_2} B t/3} \right)$$

where

$$\sigma^2 = \max\left(\sum_{t=1}^{T} w_t^2 \mathbb{E}\|V_i(t) - \Theta_i(t)\|_2^2, \left\| \sum_{t=1}^{T} w_t^2 \Sigma_i(t) \right\| \right) \le \max(n_2 B^2, 3B^2) \le 3n_2 B^2.$$

Next, using a union bound we have

$$\mathbb{P}\left(\max_{i=1,\ldots,n_1} \|k_i\|_2 \ge t \right) \le 4n_1 n_2 \exp\left(-\frac{t^2/2}{3n_2 B^2 + \sqrt{n_2} B t/3} \right).$$

Choosing $t = C\sqrt{n_2} B \log(Tn_1n_2)$ for some absolute constant C large enough in the above leads to

$$\mathbb{P}\left(\max_{i=1,\ldots,n_1} \|k_i\|_2 \ge C\sqrt{n_2} B \log(Tn_1n_2) \right) \le T^{-c}.$$

$\qquad\square$

Lemma C.3. *There exist absolute constants $c, c', c'' > 0$ such that*

$$\mathbb{P}\left(\left| \sum_{i=1}^{n_1} \tilde{U}_i^\top \tilde{V}_i - \|\tilde{\Theta}\|_F^2 \right| > C_\beta B \log(Tn_1n_2)\left(\sqrt{n_2} \|\tilde{\Theta}\|_F + B \log(Tn_1n_2) \max\{\sqrt{n_1n_2}, n_2\} \right) \right)$$
$$\le T^{-c} + T^{-c'} + 2T^{-c''}.$$

Proof. Note that $\sum_{i=1}^{n_1} \tilde{U}_i' \tilde{V}_i - \|\tilde{\Theta}\|_F^2 = I + II + III$, where

$$I = \sum_{i=1}^{n_1} (\tilde{U}_i - \tilde{\Theta}_i)^\top (\tilde{V}_i - \tilde{\Theta}_i), \quad II = \sum_{i=1}^{n_1} \tilde{\Theta}_i^\top (\tilde{V}_i - \tilde{\Theta}_i), \quad \text{and} \quad III = \sum_{i=1}^{n_1} \tilde{\Theta}_i^\top (\tilde{U}_i - \tilde{\Theta}_i).$$

It is sufficient to bound I and II, since II and III are independent copies of each other.

We start by bounding I using Lemma C.1 and Lemma C.2. Writing $k_i = \tilde{U}_i - \tilde{\Theta}_i = \sum_{t=1}^{T} w_t (U_i(t) - \Theta_i(t))$, we have from Lemma C.1 that conditional on $\{U(t)\}_{t=1}^{T}$

$$\mathbb{P}_{V|U}(|I| > \epsilon) = \mathbb{P}\left(\left| \sum_{t=1}^{T} w_t \sum_{i=1}^{n_1} k_i^\top (V_i(t) - \Theta_i(t)) \right| > \epsilon \right)$$

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$$\leq 2\exp\left(\frac{-\frac{1}{2}\epsilon^2}{3B^2\sum_{i=1}^{n_1}\|k_i\|_2^2 + \max_{i=1,\dots,n_1}\|k_i\|_2\sqrt{n_2}B2\epsilon/3}\right)$$

$$\leq 2\exp\left(\frac{-\frac{1}{2}\epsilon^2}{3n_1B^2\max_{i=1,\dots,n_1}\|k_i\|_2^2 + \max_{i=1,\dots,n_1}\|k_i\|_2\sqrt{n_2}B2\epsilon/3}\right).$$

Now by Lemma C.2, we have

$$\mathbb{P}_U\left(\max_{i=1,\dots,n_1}\|k_i\|_2 \geq C\sqrt{n_2}B\log(Tn_1n_2)\right) \leq T^{-c}.$$

Therefore, for any $\varepsilon > 0$, it holds that

$$\mathbb{P}(|I| > \varepsilon) \leq 2\exp\left(\frac{-\frac{1}{2}\epsilon^2}{3Cn_1n_2B^4\log^2(Tn_1n_2) + 2Cn_2B^2\log(Tn_1n_2)\epsilon/3}\right) + T^{-c}$$

and there exists some constant c', C' such that

$$\mathbb{P}\left(|I| > C'B^2\log^2(Tn_1n_2)\max\{\sqrt{n_1n_2}, n_2\}\right) \leq T^{-c'} + T^{-c}.$$

Now onto term II. Applying Lemma C.1 with $k_i = \tilde{\Theta}_i$ yields

$$\mathbb{P}(|II| > \varepsilon) \leq 2\exp\left(\frac{-\frac{1}{2}\epsilon^2}{3B^2\|\tilde{\Theta}\|_{\mathrm{F}} + \max_{i=1,\dots,n_1}\|\tilde{\Theta}_i\|_2\sqrt{n_2}B2\epsilon/3}\right).$$

Therefore there exist absolute constants c'', C'' such that

$$\mathbb{P}\left(|II| > C''\sqrt{n_2}B\|\tilde{\Theta}\|_{\mathrm{F}}\log(T)\right) \leq T^{-c''},$$

since $\max_{i=1,\dots,n_1}\|\tilde{\Theta}_i\|_2 = \sqrt{\max_{i=1,\dots,n_1}\|\tilde{\Theta}_i\|_2^2} \leq \|\tilde{\Theta}_i\|_{\mathrm{F}}$ and the claim follows. $\qquad\square$

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