
Unintended Selection: Persistent Qualification Rate Disparities and Interventions

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Abstract

Realistically—and equitably—modeling the dynamics of group-level disparities in machine learning remains an open problem. In particular, we desire models that do *not* suppose inherent differences between artificial groups of people—but rather endogenize disparities by appeal to unequal *initial conditions* of insular subpopulations. In this paper, agents each have a real-valued feature X (e.g., credit score) informed by a “true” binary label Y representing *qualification* (e.g., for a loan). Each agent alternately (1) receives a binary classification label \hat{Y} (e.g., loan approval) from a Bayes-optimal machine learning classifier observing X and (2) may update their qualification Y by imitating successful *strategies* (e.g., seek a raise) within an isolated group G of agents to which they belong. We consider the disparity of qualification rates $\Pr(Y = 1)$ between different groups and how this disparity changes subject to a sequence of Bayes-optimal classifiers repeatedly retrained on the global population. We model the evolving qualification rates of each subpopulation (group) using the replicator equation, which derives from a class of imitation processes. We show that differences in qualification rates between subpopulations can persist indefinitely for a set of non-trivial equilibrium states due to uniformed classifier deployments, even when groups are identical in all aspects except initial qualification densities. We next simulate the effects of commonly proposed fairness interventions on this dynamical system along with a new feedback control mechanism capable of permanently eliminating group-level qualification rate disparities. We conclude by discussing the limitations of our model and findings and by outlining potential future work.

1 Introduction

Algorithmic prediction is increasingly used for socially consequential decisions and may determine individual access to information, education, employment, credit, housing, medical treatment, freedom from incarceration, or freedom from military targeting [1–5]. This situation raises technical challenges and ethical concerns, particularly regarding the dynamics of systemic inequalities and attendant harms to society [6–8]. Nonetheless, realistically—and equitably—modeling the dynamics of disparity in machine learning remains an open problem.

Research historically considered the fairness of algorithmic predictions in terms of statistical (in)consistencies [9] (e.g., across groups [10–16] or between similar individuals [10, 11]), preference guarantees [17–19], or causal considerations [19, 20] but ignored the *response* of a population to new prediction policies. For instance, the proportions of potential loan applicants in each group that will seek higher wages, falsify income, or forego application might *change* if banks use new policies to approve or deny loans, possibly counteracting fair intent. We refer this class of fairness definitions as *normative present fairness*.

Efforts to model such population response [21–31] and the autonomous dynamical systems arising from mutual recursion with myopically updating prediction policies [21–29] have intensified, but it has remained to plausibly explain persistent disparities under group-independent prediction policies—*i.e.*, those that do not discriminate on the basis of group membership—without assuming a setting that is structurally imbalanced between groups. Our paper contributes to these efforts and considers the long-term consequences of machine learning on inter-group disparities when a sequence of classifiers induces dynamics within the rates of strategy adoption in each group. Upon adopting a dynamical framework, we note that yet another operationalization of fairness arises: the asymptotic equality of latent variables (*i.e.*, those causally responsible for outcome disparities) between groups. This notion of *long-term fairness* need not be consistent with *normative present fairness*, which may actively combat it, highlighting a tension between *ends* and *means* for fairness considerations.

1.1 Our contributions

Herein, we describe an *equitable* model of population response: one which does not suppose inherent differences between groups of people but endogenizes disparities by appeal to unequal *initial conditions*, accounting for group-specific environmental conditions as dynamical variables. We reform our notion of “groups” (*i.e.*, *subpopulations*) to appeal to natural boundaries of information exchange rather than artificially imposed classes of people. We thus offer a potentially more meaningful way to group individuals in discussions of fairness, asserting that, when considering such networks of peer exchange, “sensitive attributes” such as race, sex, color, *etc.* might not correspond to meaningful divisions of people, which depend on social context. Finally, we recognize fairness interventions as dynamical control policies that (un)intentionally select the future trajectories of a given system. We therefore allow ourselves to consider interventions that explicitly incorporate *feedback* from dynamical variables—rather than relying on fixed, prescriptive modifications of predictor loss functions.

Our first contribution is to propose a model of *classifier-induced* group-level strategy adoption that is (1) equitable, *i.e.*, free from structurally asymmetric assumptions as described above, (2) capable of explaining persistent disparities under Bayes-optimal, group-independent policies, and (3) derivable from plausible, localized information exchange between individuals. Specifically, we appeal to the replicator equation, an established model for evolutionary phenomena without mutation, to model how competing *strategies for qualification* (which determine true machine learning labels $\{0, 1\}$, affecting agent utilities) replicate within groups (*i.e.*, isolated subpopulations that differ only in size and initial proportions of qualified individuals). We ground statements with a running example involving loan applications (elaborated upon in Section 2.2) for which qualification (label $Y = 1$), interpreted as being in the public interest, implies future repayment of a loan for an applicant with feature profile X . As we avoid assuming inherent differences between groups, we consider the label-conditioned feature distribution $\Pr(X | Y = y)$ as group-independent and define qualification disparity in terms of differences in group qualification rates $\Pr(Y = 1)$. We formulate our model in Section 2, emphasizing that only the *profile of strategies* in each subpopulation is subject to evolution—narrowly qualified by the competition between strategies for replicative success—rather than the subpopulations themselves. The persistence of disparity is thus attributed to classifier policy.

Our second contribution, in Section 3, is a rigorous examination of the dynamical system formed by the replicator equation and an updating, group-independent, Bayes-optimal classifier policy, including a characterization of its equilibrium states with linear stability analysis. We identify the set of stable interior states of the system as a stable hyperplane and show that any initial state with non-zero total qualification disparity, defined in Section 3, will continue to exhibit non-zero disparity asymptotically if the state attracts to the stable hyperplane (Theorem 10). In this sense, we claim that qualification rate disparity persists indefinitely for this setting.

Our final contribution, in Section 4 is to consider a dynamics-aware fairness intervention based on feedback control that parametrically violates classifier group-independence (and therefore, in our setting, *equalized odds* [12–14] and *envy-freeness* [17, 18]) to achieve long-term fairness. We use simulation to contrast this feedback control policy to a group-independent classifier; a policy subject to demographic parity [10, 11]; and “laissez-faire”, *group-specific* policies. We conclude by discussing the limitations of our model and our findings and by outlining potential future work.

1.2 Related work

Our work chiefly contributes to the literature on fairness in machine learning but also builds on prior work on “statistical discrimination”. The most relevant publications are those that have highlighted

the importance of studying the dynamics and long-term consequences of machine learning, fairness constraints, and models of population response. In particular, Liu et al. [30] use Markov transitions to model agent responses to classification without considering classifier retraining; D’Amour et al. [22] and Zhang et al. [23] reapply Markov transitions to agent attributes in the presence of classifier retraining; Zhang et al. [32] model agents’ decisions of whether to engage with classification based on perceived accuracy and intra-group disparity; Coate and Loury [21], Hu and Chen [27], and Liu et al. [26] considered economical “best-response” models to agent labels with classifier updates; and Heidari et al. [24] considered an imitation-based model of social learning in which agents choose between the strategies of other agents to maximize utility and minimize effort. Tang et al. [33] also studied the delayed and accumulated impacts of past deployed policies, but did not study the fairness implication of such impacts. Similarly, literature on “fair bandit/reinforcement learning” [34–36] has largely focused on technical aspects of imposing normative present fairness in a sequential setting.

Our proposed model synthesizes prior conceptual innovations: First, Coate and Loury [21], Hu and Chen [27], and Ensign et al. [37] each considered incomplete information available to a classifier as a means to equitably endogenize persistent predictor bias, but did not consider incomplete information available to individual agents. Second, the class of response functions considered by Mouzannar et al. [28] allows group-level strategic responses to depend on existing qualification rates and may be used to endogenize persistent disparity under group-independent policies; the cited work does not explore this direction, but, like us, the authors assume “groups are ex-ante equal in all respect except for their qualification profiles...and any potential coupling between groups can only happen through the different and interacting selection rates induced by the policies” [28, p. 362]. Atop this foundation, we provide a plausible mechanism of imitation, motivated by incomplete information available to individual agents, to justify replicator dynamics as a special case of such response functions, and we extend a dynamical analysis for a classifier forced to contend with misclassification errors.

To support our use of the replicator equation to model group-level responses to classification, we cite the imitation-based derivation(s) of the replicator equation by Björnerstedt and Weibull [38]; the characterization of evolutionarily stable strategies conducted by Taylor and Jonker [39]; the analogy of memes as attributed to Dawkins [40]; and the extensive application of the replicator equation in game-theoretic contexts as explored by Friedman and Sinervo [41].

2 Formulation

We defer all proofs and provide them in Appendix B of the supplementary material.

We consider countably many *agents*, $n \geq 2$ *groups*, and a single *classifier*. Until Section 2.2, our setting matches that of Coate and Loury [21] but treats n groups and a more granular classifier utility function. We ground statements with a running example: a regional bank (classifier) serving several isolated communities (groups, subpopulations) by offering standardized loans for which every individual (agent) applies. Alternative examples include hiring decisions [21] or college admissions.

Agents belong to **groups**, interpreted in Section 2.2 and consistent with *isolated communities* in our running example, with known relative frequencies $\mu_g \in (0, 1)$. We vectorize these frequencies as $\boldsymbol{\mu}$.

$$\mathcal{G} := \{1, 2, \dots, n\}; \quad \forall g \in \mathcal{G}, \mu_g := \Pr(G = g); \quad \sum_{g \in \mathcal{G}} \mu_g = 1; \quad \boldsymbol{\mu} := (\mu_1, \mu_2, \dots, \mu_n) \quad (1)$$

For all statements of probability, we assign uniform probability mass to each agent.

In addition to relative size μ_g , each group has a **qualification rate** $s_g \in (0, 1)$, which we vectorize as our **state** variable \mathbf{s} . We denote the global qualification rate as \bar{s} :

$$s_g := \Pr(Y = 1 \mid G = g); \quad \mathbf{s} := (s_1, s_2, \dots, s_n); \quad \bar{s} := \sum_{g \in \mathcal{G}} \mu_g s_g = \langle \boldsymbol{\mu}, \mathbf{s} \rangle \quad (2)$$

Assumption 1. *No community is completely (un)qualified; $\forall g, s_g \in (0, 1)$.*

In our banking example, a qualified ($Y = 1$) individual will repay a loan in full if accepted ($\hat{Y} = 1$), and we presume this outcome to be desirable. The fraction of qualified individuals in community g is represented by s_g . Assumption 1 states that no community is completely (un)qualified, and, because $\mu_g \in (0, 1)$, neither is the total population, *i.e.*, $\bar{s} \in (0, 1)$.

Table 1: Agent-specific variables forming a Markov chain.

Variable	Meaning	Domain	Realizations
G	group	$\mathcal{G} = \{1, 2, \dots, n\}$	g, h, i, j
Y	qualification	$\{0, 1\}$ <i>i.e.</i> , {unqualified, qualified}	y
X	feature	$(-\infty, \infty)$	x
\hat{Y}	classification	$\{0, 1\}$ <i>i.e.</i> , {reject, accept}	\hat{y}

The **feature** X of an agent qualified as ($Y = y$) is sampled according to a probability density function q_y . In our banking example, we may interpret X as a “credit score” known to the bank.

$$q_y(x) := p_X(x | Y = y); \quad y \in \{0, 1\} \quad (3)$$

Assumption 2. *The qualification-conditioned distribution of features $q_y(x)$ is group-independent.*

Assumption 2 ensures that qualified individuals are statistically indistinguishable in terms of feature X across different communities—as are unqualified individuals. Given an agent’s qualification Y , learning G gives no additional information about X .

Assumption 3. *q_y is differentiable and strictly positive for each y . The values of X are ordered and unified such that $q_1(x)/q_0(x)$ is strictly increasing in X :*

$$\forall x, y, q_y(x) \in (0, \infty); \quad \frac{d}{dx} \left(\frac{q_1(x)}{q_0(x)} \right) > 0 \quad (4)$$

Assumption 3 ensures that the feature X is “well-behaved”: In our example, as credit scores increase as x , the odds that individuals with that credit score x will pay off loans also increases.

Finally, a **classifier** observes the feature X of each agent, from which it must predict the agent’s correct label Y using a deterministic policy π that, unless otherwise stated, remains ignorant of G .

Assumption 4. *The classifier learns the true distribution $\Pr(Y | X)$ before choosing policy π .¹*

Assumption 5. *The classifier maximizes its expected utility u with risk-neutral preferences. This utility u is linear in each outcome fraction $\Pr(Y = y, \hat{Y} = \hat{y})$, and the coefficients² $V_{y,\hat{y}} \in (-\infty, \infty)$ are independent of feature value X and group membership G . The classifier receives higher utility from correct predictions ($\hat{Y} = Y$).*

$$\hat{Y} := \pi(X); \quad u(\pi) := \sum_{y,\hat{y}=0}^1 V_{y,\hat{y}} \Pr(Y = y, \pi(X) = \hat{y}); \quad V_{y=\hat{y}} > V_{y \neq \hat{y}} \quad (5)$$

In our example, Assumption 5 is consistent with a bank maximizing expected net profit, where the bank expects net profit proportional to $V_{y,\hat{y}}$ from each individual qualified as y and approved as \hat{y} , independent of credit score X or community G . By Assumption 4, the bank selects policy π knowing the stochastic relationship between qualification Y and credit score X for the region it serves.

With group-independent classifier policies, having excised assumptions of inherent differences between groups in our formulation, we emphasize that unequal group qualification rates cause any statistical group-level disparities of prediction outcomes. We therefore consider eliminating differences in group qualification rates as a realization of long-term fairness in this setting.

Theorem 1. *Discounting sets of measure zero, the u -maximizing, group-independent policy π is parameterized by the **feature threshold** $\phi \in [-\infty, \infty]$ such that $\pi(x) = 1$ if and only if $x > \phi$, where ϕ depends only on the global qualification rate \bar{s} .*

$$\hat{y} = \pi(x) = \begin{cases} 1 & x > \phi \\ 0 & \text{otherwise} \end{cases}; \quad \frac{q_1(\phi)}{q_0(\phi)} = \xi \cdot \frac{1 - \bar{s}}{\bar{s}}; \quad \xi := \frac{V_{00} - V_{01}}{V_{11} - V_{10}} \quad (6)$$

When a solution in ϕ to the **threshold equation**, Eq. (6), does not exist, ϕ is either $\pm\infty$.

Corollary 1.1. *The classifier’s feature threshold ϕ responds inversely to \bar{s} : $\frac{d\phi}{d\bar{s}} < 0$, $\frac{d\bar{s}}{d\phi} < 0$.*

¹In practice, this distribution may be learned from sufficient data.

²We will abuse notation to write, *e.g.*, $V_{y,\hat{y}}$ as V_{10} , to disambiguate the order of indices on V and, later, U .

Assumption 6. V is such that $\xi \in (0, \infty)$ (ξ is defined in Theorem 1, Eq. (6)).

The threshold equation, Eq. (6), is a reprise of Coate and Louty [21] restricted to group-independent classifier policies. In our example, the bank maximizes its utility by approving individuals with credit scores greater than ϕ and denying everyone else. Interpreting Assumption 6, there exist populations for which the bank prefers to accept some applicants and reject others.

2.1 Time-dependence

We model our system in discrete time for semantic reasons, acknowledging that a learning process consistent with Assumption 4 requires time, although the mathematics generalize to continuous time without issue.³ Where required, we will denote time-dependence in square brackets $[t]$. Where we omit this explicit dependence, as in all prior expressions, it is understood that all variables in an expression correspond to the same time t .

Assumption 7. The relative sizes of groups μ_g , qualification-conditioned feature distributions q_y , and classifier utility coefficients $V_{y,\hat{y}}$ are all time-independent. All prior assumptions hold independently for each time step.

2.2 Replicator dynamics

Anticipating algorithmic classification, how do agents decide whether to become qualified? In our banking example, we imagine that individuals must “invest” or “apply capital” to be able to pay back loans and that the “rationality” of doing so depends on what they know about the classification policy and potential outcomes—which they must estimate from incomplete information provided by peer examples. To deal with the uncertainty of limited examples, we imagine the emergent heuristic of updating personal qualification by *imitating* the strategies of others based on popularity and “success”. For example, if your friend chose to become qualified for a loan and now runs a small business, the success of the business may induce you to seek qualification yourself by first building credit history; if many of your neighbors receive loans despite being unqualified and appear successful investing in speculative assets, you may infer that qualification is a waste of resources.

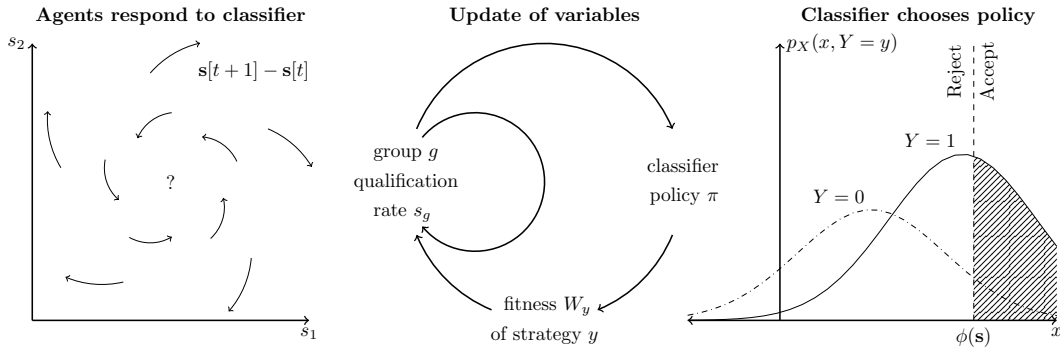


Figure 1: Our model appeals to the replicator equation Eq. (7) to model population response and considers a Bayes-optimal, group-independent classifier policy π with feature threshold ϕ (Eq. (6), right pane). When coupled (middle pane), these equations give rise to an autonomous dynamical system. We wish to understand how the vector of group qualification rates \mathbf{s} , as our state variable, changes in time (left pane).

Björnerstedt and Weibull [38] have shown that *imitation* in this form, whereby agents stochastically update to strategies weighted by success and popularity⁴ yields the (continuous time) **replicator equation**, which we use in its discrete time form, as detailed by Friedman and Sinervo [41]:

$$s_g[t + 1] = s_g[t] \frac{W_1[t]}{\bar{W}_g[t]}; \quad \bar{W}_g := W_1 s_g + W_0(1 - s_g); \quad \forall y, W_y \geq 0 \quad (7)$$

³The continuous replicator equation appears in Björnerstedt and Weibull [38] & Friedman and Sinervo [41].

⁴Weighting by popularity effects “preferential attachment” for the success of a strategy in a subpopulation and thus introduces dynamical inertia. Replicator dynamics also arise when agents update strategies with (Poisson distributed) expected periodicity that is affine in the success of one’s current strategy [38].

Here, W_y is the **fitness** of strategy y , which, by Theorem 2, we model as independent of feature X and group G . Following Björnerstedt and Weibull [38], we may derive the fitness W_y in terms of expected “success” $U_{y,\hat{y}}$ of each qualification-classification outcome (y, \hat{y}) :

Assumption 8. *The fitness of strategy $(Y = y)$, denoted as W_y^g , is affine in the average success $U_{y,\hat{y}}$ of qualification $(Y = y)$ with classification $(\hat{Y} = \hat{y})$. $U_{y,\hat{y}}$ is time-, feature- and group-independent. Without loss of generality, we restrict $U_{y,\hat{y}} \in [0, \infty)$ and drop the constant bias term from each W_y^g .*

$$W_y^g := \sum_{\hat{y}=0}^1 \Pr(\hat{Y} = \hat{y} \mid Y = y, G = g) U_{y,\hat{y}} \quad (8)$$

Theorem 2. *The fitness W_y^g of strategy $Y = y$ in group g is feature- and group-independent.*

$$\forall y, g, W_y^g = W_y \quad (9)$$

Intuitively, “success” U may be interpreted as *utility* or *payoff* to each agent when agents align strategy adoption with personal incentives, but, fundamentally, W corresponds to the relative success of the *strategy* in replicating, *i.e.*, spreading between individuals. The strategies of (non)qualification are thus subject to evolutionary pressures, competing to out-replicate each other in an environment shaped by perceptions of classifier policy. Notably, the fitness of a strategy depends only on the classifier—not group membership G . Agents remain identically modelled across all groups.

Assumption 9. *The success of (non)qualification is sensitive to classification, and the expected success for qualified individuals increases with classifier acceptance: $U_{0\hat{1}} \neq U_{0\hat{0}}$; $U_{1\hat{1}} > U_{1\hat{0}}$.*

Assumption 10. *Each group g has the properties of a closed population in which qualification, as a strategy or meme [40], competes with non-qualification free from exchange with other groups.*

It is significant that we model population updates as independent for *closed* populations, as this restricts our interpretation of groups, which must be functionally impermeable to the exchange of qualification strategies. To precisely delineate between real-world examples of “groups” is akin to disassociating “cultures”, which also imply boundaries of exchange but generally intersect. Having noted that “sensitive attributes” such as race, sex, color, *etc.* may not correspond to meaningful divisions between people (which depend on social context), we instead qualify a **group** by the extent to which it satisfies Assumption 10. As an open question, we ask whether imposing arbitrary demographic-dependent policies may catalyze the formation of groups of strategic peers, but we will consider insular social groups or isolated communities as canonical examples.

3 Dynamics

The threshold equation, Eq. (6): $\phi[t](s[t])$, and the replicator equation, Eq. (7): $s[t+1](\phi[t], s[t])$, may be coupled to yield an autonomous dynamical system $s[t+1](s[t])$ that evolves in time. To analyze it, we first generate a useful set of coordinates to compliment \bar{s} and track qualification rate disparities, defined by the differences in s_g between groups. We then note the importance of $W_1(\phi) - W_0(\phi)$ to the overall dynamics of the system, and use it to identify non-trivial equilibrium states. To interpret this section for our example, we ask how community-specific loan qualification rates change as individuals imitate successful strategies in their isolated communities, while assuming that the bank maximizes profit using group-independent credit thresholds for loan approval.

Definition 3. Define the (signed) **qualification distance** from group h to group g as

$$\delta(g, h) := s_g - s_h, \quad g, h \in \{1, 2, \dots, n\} \quad (10)$$

We next define the vector D comprising $(n - 1)$ linearly-independent qualification distances between sequential pairs of subpopulations:

$$D := \left(\delta(1, 2), \delta(2, 3), \dots, \delta(n - 1, n) \right) \quad (11)$$

The components of D and value of \bar{s} together yield a complete set of coordinates to describe the state of the dynamical system, which we may exchange for the original vector of qualification rates $\mathbf{s} = (s_1, s_2, \dots, s_n)$ via a non-orthogonal, linear change of basis (See Appendix B):

$$s_g = \bar{s} + \sum_{h=g}^{n-1} \delta(h, h+1) - \sum_{h=1}^{n-1} \sum_{k=1}^h \mu_k \delta(h, h+1) \quad \forall g \in \mathcal{G} \quad (12)$$

Let us denote the state vector in our new coordinate system as $\mathbf{r} := (\delta(1, 2), \delta(2, 3), \dots, \delta(n-1, n), \bar{s})$.

Definition 3.1. For $p \geq 1$, define a state's **p -total qualification rate disparity** as the p -norm of D :

$$\|D\|_p := \left(\sum_{g=1}^{n-1} |\delta(g, g+1)|^p \right)^{1/p} \quad (13)$$

Remark 4. States \mathbf{s} with a common \bar{s} value form a **hyperplane** $\bar{s} = \langle \boldsymbol{\mu}, \mathbf{s} \rangle$ (Eq. (2)), by definition.

Theorem 5. *The nullity of any p -total qualification rate disparity is preserved in time.*

$$p \geq 1; \quad \|D[t]\|_p = 0 \iff \|D[t+1]\|_p = 0 \quad (14)$$

Theorem 5 highlights a weak notion of the persistence of disparity within the system sans intervention: Any state that possesses some non-zero total qualification disparity (defined as some chosen p -norm of D) must always exhibit some non-zero total qualification disparity with any finite time horizon. In our example, if some communities start more qualified than others, the qualification rates of different communities will not naturally equalize in any given lifetime. Note that this statement is insufficient to address the limit $t \rightarrow \infty$, however. For a stronger result that includes this limit (Theorem 10), we first characterize the system's equilibrium states.

3.1 Equilibrium

Definition 6. The system as a whole is **at equilibrium** when, for all $g \in \mathcal{G}$ simultaneously, s_g is stationary in time:

$$\text{at equilibrium} \stackrel{\text{def}}{\iff} \forall g \in \mathcal{G}, \exists t_0 \text{ s.t. } \forall t \geq t_0, \quad s_g[t] = s_g[t_0] \quad (15)$$

Note: replicator dynamics is an instance of the more general family of monotone dynamics, with which all equilibria are shared [38, 41].

Theorem 7. *Disregarding boundary states by Assumption 1, the replicator equation, Eq. (7), implies*

$$\text{sgn}(\bar{s}[t+1] - \bar{s}[t]) = \text{sgn}(W_1(\phi[t]) - W_0(\phi[t])) \quad (16)$$

Theorem 8. *It is necessary and sufficient for a system at equilibrium that $W_1 = W_0$ or for the system to occupy some vertex of the state space.*

$$\text{at equilibrium} \iff \begin{cases} W_1 = W_0 & (\text{internal equilibrium}) \\ \forall g \in \mathcal{G}, \quad s_g \in \{0, 1\} & (\text{trivial equilibrium}) \end{cases} \quad (17)$$

Theorem 8 indicates that the conditions for internal equilibrium are described by the zeros of the function $W_1(\phi) - W_0(\phi)$, as depicted in Fig. 2, and, by the threshold equation, Eq. (6), ϕ has dynamical dependence only on \bar{s} . It follows that only certain values of \bar{s} support internal equilibrium, and each value corresponds to a hyperplane in state space (Remark 4).

Theorem 9. *$W_1(\phi) - W_0(\phi)$ is strictly quasi-concave in ϕ . This guarantees that no more than two zeros of the function $W_1 - W_0$ exist.*

We denote the possible zeros of $W_1 - W_0$ as ϕ^+ and ϕ^- , where the sign in the superscript indicates the local slope of the function. These zeroes correspond to parallel hyperplanes in state space that comprise all interior equilibria of the system. Whether ϕ^\pm corresponds to an (un)stable equilibrium hyperplane may be determined by Theorem 7 and the sign of $\frac{\partial}{\partial \phi}(W_1 - W_0)$: Only $\bar{s}(\phi^+)$ is stable, and we will verify this fact with linear stability analysis.

Theorem 10. *If the state of the system asymptotically approaches an internal equilibrium, the nullity of p -total qualification rate disparity is preserved in the limit of infinite time.*

$$p \geq 1; \quad \lim_{t' \rightarrow \infty} (W_1 - W_0) = 0 \implies \left(\|D[t]\|_p = 0 \iff \lim_{t' \rightarrow \infty} \|D[t']\|_p = 0 \right) \quad (18)$$

Theorem 10 formalizes the critical observation that any state that attracts to the stable equilibrium hyperplane, unless initially free from qualification disparity, will forever exhibit some total qualification disparity. This is a more robust notion of the **persistence of disparity** in our system than Theorem 5.

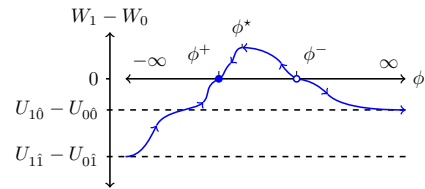


Figure 2: $W_1(\phi) - W_0(\phi)$ (blue curve) is a strictly quasi-concave function of ϕ . ϕ^* denotes the unique local extremum. The direction of the arrows is a consequence of Theorem 7 and Corollary 1.1.

3.2 Stability

For our regional banking example, we may imagine that qualification rates settle into a stable pattern in which some communities have a higher average qualification rate than others. How robust is this pattern of inequality to small fluctuations of qualification rates? Using **linear stability analysis** (*i.e.*, linearizing the response of the system to small perturbations about equilibrium and asking “do perturbations amplify or dissipate?”), we show that only the ϕ^+ -hyperplane acts as a stable attractor.

First, let us denote the evaluation of an expression *at equilibrium* by placing a vertical line to the right of the expression with “eq” as a subscript. In light of Theorem 8 and Eq. (7), we also introduce the shorthand W_{eq} to denote an equilibrium value of W_1 , W_0 , or, equivalently, any \overline{W}_g . It should be noted that the value of $W_{\text{eq}} \in [0, \infty)$ still depends on the particular equilibrium state of the system.

$$W_{\text{eq}} := W_0 \Big|_{\text{eq}} = W_1 \Big|_{\text{eq}} = \overline{W}_g \Big|_{\text{eq}} \quad \forall g \in \mathcal{G} \quad (19)$$

We linearize the system at equilibrium by constructing the Jacobian $J \in \mathbf{R}^{n \times n}$ corresponding to discrete time-evolution and identifying its eigenvectors and eigenvalues:

$$J := \begin{bmatrix} \frac{\partial \mathbf{r}}{\partial \delta(1,2)} & \frac{\partial \mathbf{r}}{\partial \delta(2,3)} & \cdots & \frac{\partial \mathbf{r}}{\partial \delta(n-1,n)} & \frac{\partial \mathbf{r}}{\partial \bar{s}} \end{bmatrix} \quad (20)$$

where \mathbf{r} , the state vector in (D, \bar{s}) coordinates, is interpreted as a column vector.

Theorem 11. *The Jacobian J simplifies to a scalar multiplied by a matrix with a single non-zero column \mathbf{v} in the last position.*

$$J \Big|_{\text{eq}} = \frac{1}{W_{\text{eq}}} \left(\frac{d\phi}{d\bar{s}} \right) \left(\frac{d}{d\phi} (W_1 - W_0) \right) \left[\mathbf{0}^{(n \times n-1)} \Big| \mathbf{v} \right], \quad \mathbf{v} := \begin{bmatrix} \delta(1,2)(1-s_1-s_2) \\ \delta(2,3)(1-s_2-s_3) \\ \cdots \\ \delta(n-1,n)(1-s_{n-1}-s_n) \\ \sum_{g \in \mathcal{G}} \mu_g s_g (1-s_g) \end{bmatrix} \quad (21)$$

The eigenvalues of J determine the stability of the system at equilibrium.

Corollary 11.1. *At equilibrium, any state displacement vector with zero \bar{s} component is an eigenvector of J with eigenvalue 0, while \mathbf{v} is an eigenvector of J with eigenvalue λ :*

$$\lambda := \left(\sum_{g \in \mathcal{G}} \mu_g s_g (1-s_g) \right) \frac{1}{W_{\text{eq}}} \left(\frac{d\phi}{d\bar{s}} \right) \left(\frac{d}{d\phi} (W_1 - W_0) \right) \Big|_{\text{eq}} \quad (22)$$

Perturbing (displacing) a state vector \mathbf{r} at an internal equilibrium by altering any combination of coordinates appearing in D —while leaving \bar{s} fixed—specifies motion on the \bar{s} hyperplane occurring in neutrally stable equilibrium (*i.e.*, a displacement vector with zero \bar{s} component has null eigenvalues at internal equilibrium. See Strogatz [42]). An internal equilibrium is stable to perturbations in \mathbf{v} , leaving the hyperplane, iff λ is negative (and, in discrete-time, > -2 to forbid over-corrections) [42].

Corollary 11.2. *As a consequence of Corollary 1.1, which states $\frac{d\phi}{d\bar{s}} < 0$, the eigenvalue λ in Eq. (22) is negative, (and the associated equilibrium hyperplane stable) iff $\frac{d}{d\phi} (W_1 - W_0) \Big|_{\text{eq}} > 0$. This prescribes precisely the value ϕ^+ for the stable equilibrium hyperplane.*

4 Interventions

In the dynamical setting we have characterized, we now explore “fairness interventions”, which substitute the set of policies that the classifier may choose from, possibly permitting group-specific decision rules π_g . We first observe that for the default policy with a group-independent feature threshold ϕ , one commonly cited standard of normative present fairness is automatically satisfied.

Definition 12. Equalized Odds [12–14] requires that a classifier’s decisions \hat{Y} , given by policy π , misclassify (un)qualified agents at equal rates across groups:

$$\forall g, h \in \mathcal{G}, \forall y, \hat{y} \in \{0, 1\}, \quad \Pr(\hat{Y} = \hat{y} \mid Y = y, G = g) = \Pr(\hat{Y} = \hat{y} \mid Y = y, G = h) \quad (23)$$

Theorem 13. *For policies defined by group-specific thresholds ϕ_g , the equivalence of these feature thresholds ($\forall g, \phi_g = \phi$) is necessary and sufficient to satisfy Equalized Odds given the group-independence of each q_y (Assumption 2).*

By Theorem 13, a group-independent policy satisfies Equalized Odds (*e.g.*, the bank accepts/rejects (un)qualified loan applicants at group-independent rates), yet disparities may persist (Theorem 10). This indicates a counter-example to reliance on Equalized Odds for long-term fairness in our model, *viz.*, the optimal group-independent threshold classifier we have studied so far.

Corollary 13.1. Equalized Odds does not imply long-term fairness in our model.

We next ask whether a small displacement a group-independent threshold ϕ near the ϕ^+ -hyperplane, which we interpret as a **universal subsidy** (or penalty), can diminish qualification rate disparities.

Theorem 14. $\Theta(\epsilon)$ perturbations of a group-independent ϕ at internal equilibrium induce motion, which, to first-order approximation (*i.e.*, ignoring $\mathcal{O}(\epsilon^2)$ terms), is parallel to the eigenvector \mathbf{v} .

As a consequence of Theorem 14, while \mathbf{v} need not be orthogonal to the equilibrium hyperplane, and a universal subsidy may decrease qualification rate disparity while applied (settling on a new equilibrium hyperplane with different, though persistent disparities), the system is stable to such perturbations at ϕ^+ as characterized by linear system response and will relax to the original equilibrium state when the intervention is removed. To permanently change qualification disparities, a temporary universal subsidy (penalty) must rely on the **non-linear response** of the system and is therefore liable to require *large* perturbations to the classifier's threshold ϕ . This finding compels us to consider interventions with group-dependent threshold perturbations—or group-dependent thresholds. To this end, we hereafter generalize our classifier such that it independently classifies each group g according to a group-specific threshold ϕ_g . We denote the vector of these thresholds as $\Phi := (\phi_1, \phi_2, \dots, \phi_n)$ and assume that, prior to some perturbative intervention, $\phi_g = \phi$ for each $g \in \mathcal{G}$.

Definition 15. Demographic Parity [10, 11] requires that a classifier's decisions \hat{Y} , given by policy π , are positive ($\hat{Y} = 1$, *e.g.*, accepting a loan application) at equal rates for all groups:

$$\forall g, h \in \mathcal{G}, \quad \Pr(\hat{Y} = 1 \mid G = g) = \Pr(\hat{Y} = 1 \mid G = h) \quad (24)$$

Definition 16. Laissez-Faire allows a separate, u -maximizing threshold ϕ_g for each group.

Theorem 17. Demographic parity requires sign-heterogeneous, group-dependent changes to the Laissez-Faire values of ϕ_g when π is non-trivial (does not uniformly accept/reject).

Satisfying demographic parity in our setting requires the solution of a differential equation in q_y (Appendix B), which we do not rigidly constrain. We therefore rely on numerical simulation, rather than analytical tools, to evaluate this intervention for our system.

Feedback control Arbitrary state transitions in the equilibrium hyperplane may be permanently effected by group-dependent perturbations to Φ , which we derive from linear system response at equilibrium. Specifically, to diminish a specific qualification distance $\delta(g, g+1)$ for given g , Φ may be perturbed by a vector quantity $\Delta_g \Phi = (\Delta_g \phi_1, \Delta_g \phi_2, \dots, \Delta_g \phi_n)$.

Theorem 18. On the stable internal equilibrium hyperplane, infinitesimal perturbation of Φ by

$$\Delta_g \Phi := -\epsilon \delta(g, g+1) \left(\frac{\alpha_g}{s_1(1-s_1)}, \dots, \frac{\alpha_g}{s_g(1-s_g)}, \frac{\beta_g}{s_{g+1}(1-s_{g+1})}, \dots, \frac{\beta_g}{s_n(1-s_n)} \right) \quad (25a)$$

$$\alpha_g := (\mu_{g+1} + \mu_{g+2} + \dots + \mu_n), \quad \beta_g := -(\mu_1 + \mu_2 + \dots + \mu_g) \quad (25b)$$

will induce motion in the system preserving \bar{s} and each $\delta(h, h+1)$ for $h \neq g$. The value of $\delta(g, g+1)$ will be diminished by a ratio proportional to the **strength parameter** $\epsilon > 0$.

Perturbations of the form $\Delta_g \Phi$ may be composed linearly for multiple values of g . In particular, when ϵ is a universal quantity, we may determine the total perturbation to Φ necessary to simultaneously and proportionately decrease all qualification distances for any given state on the stable equilibrium hyperplane. Let us denote this total perturbation as $\Delta \Phi := \sum_{g \in \mathcal{G}} \Delta_g \Phi = (\Delta \phi_1, \Delta \phi_2, \dots, \Delta \phi_n)$. Component-wise, $\Delta \Phi$ is given by

$$\Delta \phi_g = \frac{-\epsilon}{s_g(1-s_g)} \left(\sum_{h=g}^{n-1} \alpha_h \delta(h, h+1) + \sum_{h=1}^{g-1} \beta_h \delta(h, h+1) \right) \quad (26)$$

We note that the proposed feedback control mechanism depends only on the known constants μ_g and feedback in terms of current qualification distances δ . In addition, the force of the intervention can be tuned by setting the strength parameter ϵ . Finally, we remark that this mechanism can be composed with global perturbations of ϕ , *i.e.*, universal subsidy in the manner of Theorem 14, to intervene without rejecting any agents that would have been accepted under a group-independent policy.

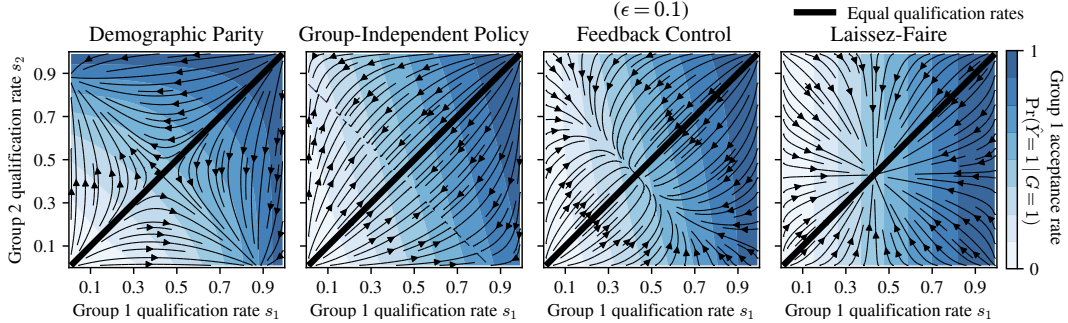


Figure 3: Simulated dynamics for two groups of equal size, subject to different global interventions. Streamlines approximate system time evolution. q_0 and q_1 are Gaussians with unit variance and have means -1 and 1 , respectively. Other examples and rendered dynamical variables are provided in Appendix C. For this example, $(U_{0\hat{0}} = 0.1; U_{0\hat{1}} = 5.5; U_{1\hat{0}} = 0.5; U_{1\hat{1}} = 1.0; V_{0\hat{0}} = 0.5; V_{0\hat{1}} = -0.5; V_{1\hat{0}} = -0.25; V_{1\hat{1}} = 1.0)$.

We compare interventions by appeal to simulation, choosing a setting that guarantees a single, stable average qualification rate \bar{s}^* under group-independent policies (**GI**) (Fig. 3). We consider trade-offs between *normative present fairness* (\mathbf{F}_{NP}) (e.g., demographic parity (**DP**) or equalized odds (**EO**)) and *long-term fairness* (\mathbf{F}_{LT}), for which the dynamics must converge to the line demarcating equal qualification rates. Darker shading (blue) implies a higher absolute acceptance rate for Group 1, which, by the setting’s symmetry, is the same for Group 2 when reflected across the aforementioned line; reflectional asymmetry violates DP. Under compulsory DP (first pane), the system violates \mathbf{F}_{LT} , settling into a “patronizing equilibrium under affirmative action”, as coined by Coate and Loury [21], in which agents from a less-qualified group are *patronized* (e.g., granted loans despite nonqualification) by the classifier (cf. the upper-left corner, with low qualification and high acceptance rates for Group 1). States under GI (second pane), which satisfies the EO notion of \mathbf{F}_{NP} by Theorem 13, converge to a line of constant \bar{s} (cf. Remark 4) while preserving qualification disparities (cf. Theorem 10). \mathbf{F}_{LT} is expected from a **laissez-faire** (**LZ**) policy (last pane), which adopts group-specific policies and thus decouples all group dynamics: Each group must converge to \bar{s}^* separately. Still, LZ satisfies neither the DP (by reflectional asymmetry) nor EO (by Theorem 13) notions of \mathbf{F}_{NP} . In contrast, feedback control (third pane) achieves \mathbf{F}_{LT} by conceding ϵ -small, parametric violations of \mathbf{F}_{NP} (EO) (See Appendix C for plots of classifier error rates in this setting).

5 Discussion and limitations

The novelty of our contribution is the demonstration of persistent qualification rate disparities in a symmetric setting consistent with plausible mechanisms of population response—sustained by the careless deployment of machine learning and myopic fairness interventions. We submit that, given the many charitable assumptions of our model to achieve perfect structural equality between groups, any reasonable fairness intervention should succeed in responsibly rectifying disparities here, if anywhere. Moreover, we have laid bare inherent tensions that can exist between the *means* and *ends* of fairness considerations in a dynamical context, demonstrating the potential incompatibility of immediate and long-term notions of fairness.

We acknowledge that our model is simplistic, but such simple cases must be well-understood as a first step towards further, equitable models of population response. We regard the requirement of strictly isolated groups as the most tenuous assumption of our model and conjecture that even relatively weak inter-group exchange of strategies should lead to long-term fairness in our default setting. Nonetheless, we believe that a program based on incomplete agent information can successfully endogenize persistent disparities in symmetric settings more robustly. Specifically, future work may consider multiple classifiers with different task domains affecting a common population; we expect this extension to readily endogenize broken symmetries between group environments and conditions. We also trust that voluntary participation, as considered by Zhang et al. [32], may be modelled as an additional strategy within our framework. Regarding empirical falsifiability, we note that the dynamics of social disparity are not exclusive to algorithmic classifiers [27], and ask whether our model’s predictions may be contrasted with existing and historical resource allocation problems.

We invite readers to consider both our model and application of control theory to society through algorithmic classification, with care. We intend our work to *reform* the misapplication of machine learning, inappropriate modelling assumptions, and myopic notions of fairness.

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Supplementary Material

Unintended Selection: Persistent Qualification Rate Disparities and Interventions

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The Appendices are organized according to Table 2.

Table 2: Organization of appendices

Appendix	Content
Appendix A	Summary of notation used in the main paper
Appendix B	Proofs of all theorems, remarks, and corollaries as well as additional lemmas
Appendix C	Figures like Fig. 3, exploring different settings and variables of interest

A Notation

Table 3: Choice of notation

<u>Parameters:</u>	
n	Number of groups
\mathcal{G}	Set of groups $\{1, 2, \dots, n\}$
μ_g	Fraction of total population in group g
$\boldsymbol{\mu}$	Vector $(\mu_1, \mu_2, \dots, \mu_n)$
V	2×2 matrix of classifier utilities, indexed by (Y, \hat{Y}) pairs
θ	Classifier's probability threshold (Lemma 1.1)
U	2×2 matrix of agent fitnesses, indexed by (Y, \hat{Y}) pairs
q_y	Probability density function of X given $Y = y$
Q_y	Cumulative distribution function of X given $Y = y$
<u>Random Variables:</u>	
G	Group to which an agent belongs
X	Real-valued feature of an agent
Y	Actual binary label (qualification) of an agent
\hat{Y}	Predicted binary label (qualification) of an agent
<u>Indices:</u>	
g, h, i, j	used to indicate a group
x	used to indicate a feature value
y	used to indicate a binary label (qualification)
\hat{y}	used to indicate a predicted binary label (qualification)
<u>Dynamical Variables:</u>	
t	Discrete time
$\cdot[t]$	Restriction of a dynamical variable to time t
s_g	Fraction of qualified agents in group g
\mathbf{s}	Vector (s_1, s_2, \dots, s_n)
\bar{s}	Fraction of qualified agents in total population
$\delta(g, h)$	Difference in qualification rates between groups: $s_g - s_h$
D	A set of $n - 1$ linearly independent qualification distances δ
\mathbf{r}	Vector with the elements of D prepending \bar{s} as components
π	Classifier's policy mapping X to \hat{Y}
ϕ	Classifier's feature threshold (Theorem 1)
\overline{W}_y	Average agent fitness conditioned on $Y = y$
\overline{W}_g	Average agent fitness conditioned on $G = g$
<u>Miscellaneous:</u>	
J	Jacobian matrix for dynamical system.
λ	A specific eigenvalue of J
\mathbf{v}	A specific eigenvector of J
Φ	Vector of group-specific feature thresholds ϕ_g

B Proofs

Proof of Theorem 1

Lemma 1.1. *Discounting sets of measure zero, the u -maximizing policy π is parameterized by a single **probability threshold** $\theta \in [0, 1]$ such that*

$$\pi(x) = \begin{cases} 1 & \Pr(Y = 1 \mid X = x) > \theta \\ 0 & \text{otherwise} \end{cases} \quad (27)$$

where

$$\theta = \frac{V_{0\hat{0}} - V_{0\hat{1}}}{V_{1\hat{1}} - V_{0\hat{1}} + V_{0\hat{0}} - V_{0\hat{1}}} \quad (28)$$

Proof of Lemma 1.1. Discounting sets of measure zero (i.e., rejecting the possibility of strict equality as infinitely unlikely), a classifier will accept an agent with feature value $X = x$ if and only if the expected utility of doing so is greater than rejecting.

$$\left(\sum_{y=0}^1 \Pr(Y = y \mid X = x) V_{y\hat{1}} \right) > \left(\sum_{y=0}^1 \Pr(Y = y \mid X = x) V_{y\hat{0}} \right) \quad (29)$$

This reduces algebraically to

$$\frac{\Pr(Y = 1 \mid X = x)}{\Pr(Y = 0 \mid X = x)} > \xi \quad (30)$$

where

$$\xi := \frac{V_{0\hat{0}} - V_{0\hat{1}}}{V_{1\hat{1}} - V_{1\hat{0}}} \quad (31)$$

and by change of variables

$$\theta := \frac{\xi}{1 + \xi} = \frac{V_{0\hat{0}} - V_{0\hat{1}}}{V_{1\hat{1}} - V_{1\hat{0}} + V_{0\hat{0}} - V_{0\hat{1}}}, \quad \xi = \frac{\theta}{1 - \theta} \quad (32)$$

to

$$\Pr(Y = 1 \mid X = x) > \theta \quad (33)$$

Eq. (33) is thus the sole criterion for accepting an agent with feature value $X = x$, and our proof is complete.

Lemma 1.2.

$$\Pr(Y = 1 \mid X = x) = \frac{\bar{s}q_1(x)}{\bar{s}q_1(x) + (1 - \bar{s})q_0(x)} \quad (34)$$

Proof of Lemma 1.2. By Bayes's Theorem,

$$\Pr(G = g, Y = 1 \mid X = x) = \frac{p_X(x \mid G = g, Y = 1) \Pr(Y = 1 \mid G = g) \Pr(G = g)}{p_X(x)} \quad (35a)$$

$$= \frac{q_1(x) s_g \mu_g}{\sum_{h \in \mathcal{G}} (s_h q_1(x) + (1 - s_h) q_0(x)) \mu_h} \quad (35b)$$

By marginalizing over groups g , it follows that

$$\Pr(Y = 1 \mid X = x) = \frac{\sum_{g \in \mathcal{G}} s_g q_1(x) \mu_g}{\sum_{h \in \mathcal{G}} (s_h q_1(x) + (1 - s_h) q_0(x)) \mu_h} \quad (36)$$

This expression may be simplified to the target statement by substituting from Eq. (1):

$$\bar{s} := \sum_{g \in \mathcal{G}} \mu_g s_g \quad (37)$$

$$(1 - \bar{s}) = \sum_g \mu_g - \sum_{g \in \mathcal{G}} \mu_g s_g \quad (38a)$$

$$= \sum_{g \in \mathcal{G}} \mu_g (1 - s_g) \quad (38b)$$

Lemma 1.3. $\Pr(Y = 1 \mid X = x)$ and $\Pr(Y = 0 \mid X = x)$ have support for all x .

$$\forall x \in (-\infty, \infty), 0 < \Pr(Y = 1 \mid X = x) < 1 \quad (39)$$

Proof of Lemma 1.3. By Assumption 3 and Assumption 1, $\bar{s}q_1(x)$ and $(1 - \bar{s})q_0(x)$ must both be strictly positive. As an immediate consequence of Lemma 1.2 and the identity $\Pr(Y = 1 \mid X = x) = 1 - \Pr(Y = 0 \mid X = x)$, we conclude that both $\Pr(Y = 1 \mid X = x)$ and $\Pr(Y = 0 \mid X = x)$ are greater than zero.

Lemma 1.4. $\Pr(Y = 1 \mid X = x)$ is monotonically increasing in X .

Proof of Lemma 1.4. By Assumption 3, we have that

$$\forall y, x, \quad q_y(x) \in (0, \infty) \text{ and } \frac{d}{dx} \left(\frac{q_1(x)}{q_0(x)} \right) > 0$$

While from Lemma 1.2,

$$\Pr(Y = 1 \mid X = x) = \frac{\bar{s}q_1(x)}{\bar{s}q_1(x) + (1 - \bar{s})q_0(x)}$$

By the differentiability and strict positivity of each q (Assumption 3) as well as the strict positivity of $\Pr(Y = 1 \mid X = x)$ by Lemma 1.3, it is sufficient to show strict positivity of the first derivative of $\Pr(Y = 1 \mid X = x)$ to prove monotonicity. We therefore wish to show

$$\frac{d}{dx} \left(\frac{q_1(x)}{q_0(x)} \right) > 0 \implies \frac{d}{dx} \left(\frac{\bar{s}q_1(x)}{\bar{s}q_1(x) + (1 - \bar{s})q_0(x)} \right) > 0 \quad (40)$$

First, let us define

$$a(x) := \bar{s}q_1(x); \quad b(x) := (1 - \bar{s})q_0(x) \quad (41)$$

Our objective may therefore be rewritten as

$$\left(\frac{1 - \bar{s}}{\bar{s}} \right) \frac{d}{dx} \left(\frac{a(x)}{b(x)} \right) > 0 \implies \frac{d}{dx} \left(\frac{a(x)}{a(x) + b(x)} \right) > 0 \quad (42)$$

Performing explicit differentiation,

$$\frac{d}{dx} \left(\frac{a(x)}{b(x)} \right) = \left(\frac{a(x) + b(x)}{b(x)} \right)^2 \frac{d}{dx} \left(\frac{a(x)}{a(x) + b(x)} \right) \quad (43)$$

we again rewrite our objective as

$$\frac{1 - \bar{s}}{\bar{s}} \left(\frac{a(x) + b(x)}{b(x)} \right)^2 \frac{d}{dx} \left(\frac{a(x)}{a(x) + b(x)} \right) > 0 \implies \frac{d}{dx} \left(\frac{a(x)}{a(x) + b(x)} \right) > 0 \quad (44)$$

since $\frac{1 - \bar{s}}{\bar{s}} > 0$ by Assumption 1, this is a necessarily true statement.

Theorem 1 Statement. Discounting sets of measure zero, the u -maximizing, group-independent policy π is parameterized by the feature threshold $\phi \in [-\infty, \infty]$ such that $\pi(x) = 1$ if and only if $x > \phi$, where ϕ depends only on the global qualification rate \bar{s} as

$$\xi := \frac{V_{0\hat{0}} - V_{0\hat{1}}}{V_{1\hat{1}} - V_{1\hat{0}}}; \quad \frac{q_1(\phi)}{q_0(\phi)} = \xi \left(\frac{1 - \bar{s}}{\bar{s}} \right) \quad (45)$$

When a solution in ϕ to the threshold equation, Eq. (45), does not exist, ϕ is either $\pm\infty$.

Proof of Theorem 1. By Lemma 1.1, a threshold value for $\Pr(Y = 1 | X = x)$ (i.e., θ) is sufficient to characterize the optimal classifier policy π :

$$\pi(x) = \begin{cases} 1 & \Pr(Y = 1 | X = x) > \theta \\ 0 & \text{otherwise} \end{cases} \quad (46)$$

Lemma 1.4 concludes that $\Pr(Y = 1 | X = x)$ is monotonic in X , and so it follows that whenever $\Pr(Y = 1 | X = x)$ achieves the threshold value of θ , it does so at a corresponding, unique feature-threshold value $X = \phi$

$$\forall x, \Pr(Y = 1 | X = x) = \theta \implies x = \phi \quad (47)$$

such that the classifier's policy is given by

$$\pi(x) = \begin{cases} 1 & x > \phi \\ 0 & \text{otherwise} \end{cases} \quad (48)$$

When such equality between $\Pr(Y = 1 | X = x)$ and θ never occurs, we are free to define

$$\phi = \begin{cases} -\infty & \min_x \{ \Pr(Y = 1 | X = x) \} > \theta \\ \infty & \max_x \{ \Pr(Y = 1 | X = x) \} < \theta \end{cases} \quad (49)$$

so that Eq. (48) remains valid in all cases.

Finally, when equality between $\Pr(Y = 1 | X = x)$ and θ does occur, we may solve for finite ϕ by re-expressing $\Pr(Y = 1 | X = x)$ according to Lemma 1.2:

$$\theta = \frac{\bar{s}q_1(\phi)}{\bar{s}q_1(\phi) + (1 - \bar{s})q_0(\phi)} \quad (50)$$

Algebraic manipulations are sufficient to derive Eq. (45), where we appeal to Assumption 1 ($s_g \in (0, 1)$) and Assumption 6 ($\xi \in (0, \infty)$), thus $\theta \in (0, 1)$ by Eq. (32) to ensure that we do not divide by 0.

Proof of Corollary 1.1

Corollary 1.1 Statement. The classifier's feature threshold ϕ responds inversely to \bar{s} :

$$\frac{d\phi}{d\bar{s}} < 0, \quad \frac{d\bar{s}}{d\phi} < 0 \quad (51)$$

Proof of Corollary 1.1. Let us differentiate Eq. (45) with respect to \bar{s} . By the chain rule,

$$\frac{d}{d\phi} \left(\frac{q_1(\phi)}{q_0(\phi)} \right) \left(\frac{d\phi}{d\bar{s}} \right) = \left(\frac{\theta}{1 - \theta} \right) \frac{d}{d\bar{s}} \left(\frac{1 - \bar{s}}{\bar{s}} \right) \quad (52a)$$

$$= \left(\frac{\theta}{1 - \theta} \right) \left(\frac{-1}{(\bar{s})^2} \right) \quad (52b)$$

By Assumption 3, Assumption 6, we observe

$$\frac{d}{d\phi} \left(\frac{q_1(\phi)}{q_0(\phi)} \right) > 0; \quad \left(\frac{\theta}{1 - \theta} \right) > 0; \quad \left(\frac{-1}{(\bar{s})^2} \right) < 0 \quad (53)$$

Therefore, by accounting for the sign of each factor in Eq. (52b) and the relationship between derivatives of inverse functions, we conclude that

$$\frac{d\phi}{d\bar{s}} < 0, \quad \frac{d\bar{s}}{d\phi} < 0 \quad (54)$$

Proof of Theorem 2

Definition 2.1. Define the **cumulative distribution functions** Q_y such that

$$Q_y(\phi) := \int_{-\infty}^{\phi} q_y(x) dx, \quad y \in \{0, 1\} \quad (55)$$

Lemma 2.1. $Q_y(\phi)$ is group-independent.

$$Q_y(\phi) = \Pr(\hat{Y} = 0 \mid Y = y) \quad (56)$$

Proof of Lemma 2.1.

$$Q_i(\phi) := \int_{-\infty}^{\phi} q_i(x) dx \quad (57a)$$

$$= \int_{\{x: \pi(x)=0\}} p_X(x \mid Y = i) \quad (57b)$$

$$= \int_{\{x: \pi(x)=0\}} p_X(x \mid Y = i, G = g), \quad \forall g \in \mathcal{G} \quad (57c)$$

$$= \Pr(\hat{Y} = 0 \mid Y = i, G = g), \quad \forall g \in \mathcal{G} \quad (57d)$$

$$= \Pr(\hat{Y} = 0 \mid Y = y) \quad (57e)$$

where Eq. (57b) is a consequence of Theorem 1, and Eq. (57c) follows from Assumption 2.

Theorem 2 Statement. The fitness W_y^g of strategy $Y = y$ in group g is feature- and group-independent.

$$\forall y, g, W_y^g = W_y \quad (58)$$

Proof of Theorem 2. By Assumption 8 and Lemma 2.1,

$$W_y^g = U_{y\hat{1}} + (U_{y\hat{0}} - U_{y\hat{1}})Q_y(\phi), \quad y \in \{0, 1\} \quad (59)$$

This expression is also group-independent, and we may denote

$$\forall y, g, W_y^g = W_y \quad (60)$$

Verification of Eq. (12)

Eq. (12) Statement.

$$s_g = \bar{s} + \sum_{h=g}^{n-1} \delta(h, h+1) - \sum_{h=1}^{n-1} \sum_{k=1}^h \mu_k \delta(h, h+1)$$

Verification of Eq. (12). We will verify from Eq. (12) directly that

$$\sum_g s_g \mu_g = \bar{s}; \quad s_g - s_{g+1} = \delta(g, g+1) \quad (61)$$

First, let us verify that $\sum_g s_g \mu_g = \bar{s}$, recalling $\sum_{g=1}^n \mu_g = 1$. We recognize that the first and third unexpanded terms in Eq. (12) are unvarying with g , while the second term, when summed, negates the third. That is, despite prescribing a different order of summation, precisely the same values are summed in

$$\sum_{g=1}^n \mu_g \sum_{h=g}^{n-1} \delta(h, h+1) = \sum_{h=1}^{n-1} \sum_{k=1}^h \mu_k \delta(h, h+1) = \sum_{1 \leq i \leq j < n} \mu_i \delta(j, j+1) \quad (62)$$

and so, as desired,

$$\sum_{g=1}^n \mu_g s_g = \bar{s} \quad (63)$$

Next, we rewrite the definitions of α_g and β_g appearing in Theorem 18

$$\alpha_g := \sum_{h=g+1}^n \mu_h = (\mu_{g+1} + \mu_{g+2} + \dots + \mu_n) \quad (64)$$

$$\beta_g := -\sum_{h=1}^g \mu_h = -(\mu_1 + \mu_2 + \dots + \mu_g) \quad (65)$$

We note that $\alpha_g - \beta_g = 1$ and may rewrite the change of coordinates given in Eq. (12) as

$$s_g = \bar{s} + \sum_{h=g}^{n-1} \delta(h, h+1) - \sum_{h=1}^{n-1} \sum_{k=1}^h \mu_k \delta(h, h+1) \quad (66a)$$

$$= \bar{s} + \sum_{h=g}^{n-1} \delta(h, h+1) + \sum_{h=1}^{n-1} \beta_h \delta(h, h+1) \quad (66b)$$

$$= \bar{s} + \sum_{h=g}^{n-1} \alpha_h \delta(h, h+1) + \sum_{h=1}^{g-1} \beta_h \delta(h, h+1) \quad (66c)$$

from which we may verify that

$$s_g - s_{g+1} = \alpha_g \delta(g, g+1) - \beta_g \delta(g, g+1) = \delta(g, g+1) \quad (67)$$

It follows that Eq. (12) inverts the linear change of coordinates prescribed by the definitions of \bar{s} and $\delta(g, g+1)$.

Proof of Remark 4

Remark 4 Statement. States \mathbf{s} with a common \bar{s} value form a **hyperplane** $\bar{s} = \langle \boldsymbol{\mu}, \mathbf{s} \rangle$ (Eq. (2)), by definition.

Proof of Remark 4. This follows from the definition of a hyperplane and Eq. (2)

$$\bar{s} := \sum_{g \in \mathcal{G}} \mu_g s_g = \langle \boldsymbol{\mu}, \mathbf{s} \rangle \quad (68)$$

Proof of Theorem 5

Theorem 5 Statement. The nullity of any p -total qualification rate disparity is preserved in time.

$$p \geq 1; \quad \|D[t]\|_p = 0 \iff \|D[t+1]\|_p = 0 \quad (69)$$

Proof of Theorem 5. We first prove the forward direction (for any p), $\mathcal{H} = \{1, 2, \dots, n-1\}$

$$\sum_{g=1}^{n-1} \|\delta(g, g+1)[t]\|_p = 0 \implies \delta(h, h+1)[t] = 0 \quad \forall h \in \mathcal{H} \quad (70a)$$

$$\implies s_h[t] = s_{h+1}[t], \quad \overline{W}_h[t] = \overline{W}_{h+1}[t] \quad \forall h \in \mathcal{H} \quad (70b)$$

$$\implies s_h[t] \frac{W_1}{\overline{W}_h[t]} = s_{h+1}[t] \frac{W_1}{\overline{W}_{h+1}[t]} \quad \forall h \in \mathcal{H} \quad (70c)$$

$$\implies s_h[t+1] = s_{h+1}[t+1] \quad \forall h \in \mathcal{H} \quad (70d)$$

$$\implies \delta(h, h+1)[t+1] = 0 \quad \forall h \in \mathcal{H} \quad (70e)$$

$$\implies \sum_{g=1}^{n-1} \|\delta(g, g+1)[t+1]\|_p = 0 \quad (70f)$$

Next, we prove the reverse direction (for any p):

$$\sum_{g=1}^{n-1} \|\delta(g, g+1)[t+1]\|_p = 0 \implies \delta(h, h+1)[t+1] = 0 \quad \forall h \in \mathcal{H} \quad (71a)$$

$$\implies s_h[t+1] = s_{h+1}[t+1] \quad \forall h \in \mathcal{H} \quad (71b)$$

$$\implies s_h[t] \frac{W_1}{\overline{W}_h[t]} = s_{h+1}[t] \frac{W_1}{\overline{W}_{h+1}[t]} \quad \forall h \in \mathcal{H} \quad (71c)$$

$$\implies s_h[t] = s_{h+1}[t] \quad \forall h \in \mathcal{H} \quad (71d)$$

$$\implies \delta(h, h+1)[t] = 0 \quad \forall h \in \mathcal{H} \quad (71e)$$

$$\implies \sum_{g=1}^{n-1} \|\delta(g, g+1)[t]\|_p = 0 \quad (71f)$$

where Eq. (71d) follows from

$$\frac{s_g}{s_g W_1 + (1-s_g)W_0} = \frac{s_h}{s_h W_1 + (1-s_h)W_0} \quad (72a)$$

$$\implies s_g(s_h W_1 + (1-s_h)W_0) = s_h(s_g W_1 + (1-s_g)W_0) \quad (72b)$$

$$\implies W_0 s_g = W_0 s_h \quad (72c)$$

$$\implies s_g = s_h \quad (72d)$$

Proof of Theorem 7

Theorem 7 Statement.

Disregarding boundary states by Assumption 1, the replicator equation, Eq. (7), implies

$$\text{sgn}(\overline{s}[t+1] - \overline{s}[t]) = \text{sgn}(W_1(\phi[t]) - W_0(\phi[t])) \quad (73)$$

Proof of Theorem 7. There are three mutually exclusive cases we must consider by appeal to the replicator equation (Eq. (7)) and Assumption 1 ($s_g \in (0, 1)$ for all g in \mathcal{G}). Specifically, we verify that

$$W_1 > W_0 \implies \forall g \in \mathcal{G}, \frac{W_1}{\overline{W}_g} > 1 \implies \overline{s}[t+1] > \overline{s}[t] \quad (74a)$$

$$W_1 = W_0 \implies \forall g \in \mathcal{G}, \frac{W_1}{\overline{W}_g} = 1 \implies \overline{s}[t+1] = \overline{s}[t] \quad (74b)$$

$$W_1 < W_0 \implies \forall g \in \mathcal{G}, \frac{W_1}{\overline{W}_g} < 1 \implies \overline{s}[t+1] < \overline{s}[t] \quad (74c)$$

The sign of $W_1 - W_0$ therefore determines the sign of $\overline{s}[t+1] - \overline{s}[t]$ directly. Likewise, the sign of $\overline{s}[t+1] - \overline{s}[t]$ implies the sign of $W_1 - W_0$, for any discrepancy would imply a contradiction:

$$\overline{s}[t+1] > \overline{s}[t] \implies W_1 > W_0 \quad (75a)$$

$$\overline{s}[t+1] = \overline{s}[t] \implies W_1 = W_0 \quad (75b)$$

$$\overline{s}[t+1] < \overline{s}[t] \implies W_1 < W_0 \quad (75c)$$

Proof of Theorem 8

Theorem 8 Statement.

It is necessary and sufficient for a system at equilibrium that $W_1 = W_0$ or for the system to occupy some vertex of the state space.

$$\text{at equilibrium} \iff \begin{cases} W_1 = W_0 & \text{(internal equilibrium)} \\ \forall g \in \mathcal{G}, s_g \in \{0, 1\} & \text{(trivial equilibrium)} \end{cases} \quad (76)$$

Proof of Theorem 8. By the definition of equilibrium (Definition 6), the forward direction implies equality between $s_g[t]$ and $s_g[t_0 + 1]$ for all $g \in \mathcal{G}$. It follows from the replicator equation (Eq. (7)) that at least one condition must be met to guarantee this equality, and at least one is consequent:

$$\forall g \in \mathcal{G}, \quad (77a)$$

$$s_g[t_0 + 1] = s_g[t_0] \iff \left(\frac{W_1}{W_1 s_g + W_0(1 - s_g)}[t_0] = 1 \vee s_g[t_0] = 0 \right) \quad (77b)$$

We consider the first case further and note that it is true if and only if $W_1 = W_0$ or $s_g = 1$.

$$\frac{W_1}{W_1 s_g + W_0(1 - s_g)} = 1 \iff W_1 = W_1 s_g + W_0(1 - s_g) \quad (78a)$$

$$\iff (1 - s_g)W_1 = W_0(1 - s_g) \quad (78b)$$

$$\iff \left(W_1 = W_0 \vee s_g = 1 \right) \quad (78c)$$

Thus the tree of cases terminates with, for each $g \in \mathcal{G}$, any one of $s_g = 0$, $s_g = 1$, or $W_0 = W_1$, as necessary and sufficient for the state to be at equilibrium. We note that if $W_0 = W_1$, no other conditions must be considered separately for different values of g . We may therefore re-express these conditions for equilibrium succinctly as the two cases we set out to show:

$$\text{at equilibrium} \iff \left(W_1 = W_0 \vee \forall g \in \mathcal{G}, s_g \in \{0, 1\} \right) \quad (79)$$

Note: It is possible that $W_1 = W_0$ at some trivial equilibrium, and so our use of the term internal equilibrium does not strictly limit us to consideration of states in the interior of the state space (i.e., points removed from the boundary).

Proof of Theorem 9

Theorem 9 Statement. $W_1(\phi) - W_0(\phi)$ is strictly quasi-concave in ϕ . This guarantees that no more than two zeros of the function $W_1 - W_0$ exist.

Proof of Theorem 9. We proceed by characterizing the function $W_1(\phi) - W_0(\phi)$, starting with its zeros. By Theorem 2 and Lemma 2.1, the values of ϕ for which $W_1(\phi) - W_0(\phi) = 0$ must satisfy

$$\left(U_{1\hat{1}} + (U_{1\hat{0}} - U_{1\hat{1}})Q_1(\phi) \right) - \left(U_{0\hat{1}} + (U_{0\hat{0}} - U_{0\hat{1}})Q_0(\phi) \right) = 0 \quad (80)$$

Next, we consider the first derivative of $W_1 - W_0$ with respect to ϕ :

$$\frac{d}{d\phi} \left(W_1(\phi) - W_0(\phi) \right) = q_1(\phi)(U_{1\hat{0}} - U_{1\hat{1}}) - q_0(\phi)(U_{0\hat{0}} - U_{0\hat{1}}) \quad (81a)$$

$$= \left(\frac{q_1(\phi)}{q_0(\phi)} - \frac{U_{0\hat{0}} - U_{0\hat{1}}}{U_{1\hat{0}} - U_{1\hat{1}}} \right) \left(q_0(\phi)(U_{1\hat{0}} - U_{1\hat{1}}) \right) \quad (81b)$$

Recall that $U_{i\hat{0}} \neq U_{i\hat{1}}$ and $U_{1\hat{0}} < U_{1\hat{1}}$ (Assumption 9). By the strict (increasing) monotonicity of $\frac{q_1(\phi)}{q_0(\phi)}$ in ϕ and strict positivity of $q_0(\phi)$, both guaranteed by Assumption 3, the sign of this expression can change at most once as ϕ is varied from $-\infty$ to ∞ . We denote the value of ϕ at which the sign of this first derivative changes as ϕ^* :

$$\frac{q_1(\phi^*)}{q_0(\phi^*)} = \frac{U_{0\hat{0}} - U_{0\hat{1}}}{U_{1\hat{0}} - U_{1\hat{1}}} \quad (82)$$

Moreover, it follows that

$$\phi < \phi^* \implies \frac{d}{d\phi} W_1 - W_0 > 0 \quad (83a)$$

$$\phi > \phi^* \implies \frac{d}{d\phi} W_1 - W_0 < 0 \quad (83b)$$

$W_1 - W_0$ is therefore strictly quasi-concave, from which it follows that only two zeros of the function can exist (*By contradiction, more than two zeros would require the function, which has no discontinuities, to invert its slope more than once.*)

For completeness, we may also take a second derivative of $W_1 - W_0$ with respect to ϕ :

$$\begin{aligned} \frac{d^2}{d\phi^2} (W_1(\phi) - W_0(\phi)) &= \frac{d}{d\phi} \left(\frac{q_1(\phi)}{q_0(\phi)} \right) \left(q_0(\phi)(U_{1\hat{0}} - U_{1\hat{1}}) \right) \\ &+ \left(\frac{q_1(\phi)}{q_0(\phi)} - \frac{U_{0\hat{0}} - U_{0\hat{1}}}{U_{1\hat{0}} - U_{1\hat{1}}} \right) \left(\frac{d}{d\phi} q_0(\phi) \right) \end{aligned} \quad (84)$$

Doing so, we observe that $W_1 - W_0$ may have any number of inflection points, but ϕ^* cannot be one of them. We see this because the second term of the expression above evaluated at ϕ^* must be zero, but the first term must be non-zero by Assumption 3 and Assumption 9. It follows that ϕ^* is the unique occurrence of a local extremum and therefore a global extremum of $W_1 - W_0$.

Proof of Theorem 10

Theorem 10 Statement. If the state of the system asymptotically approaches an internal equilibrium, the nullity of p -total qualification rate disparity is preserved in the limit of infinite time.

$$p \geq 1; \quad \lim_{t' \rightarrow \infty} (W_1 - W_0) = 0 \implies \left(\|D[t]\|_p = 0 \iff \lim_{t' \rightarrow \infty} \|D[t']\|_p = 0 \right) \quad (85)$$

Proof of Theorem 10. Assuming that a state asymptotically approaches the equilibrium hyperplane ($\lim_{t' \rightarrow \infty} (W_1 - W_0) = 0$), Let us first prove the forward direction of the desired mutual implication. If a state starts with zero total qualification rate disparity, it follows by Theorem 5 that the total qualification rate disparity remains zero for all time, and so

$$p \geq 1; \quad \lim_{t' \rightarrow \infty} (W_1 - W_0) = 0 \implies \left(\|D[t]\|_p = 0 \implies \lim_{t' \rightarrow \infty} \|D[t']\|_p = 0 \right) \quad (86)$$

For the reverse direction, let us define \mathbf{s}^* as the unique disparity-free state on the stable internal equilibrium hyperplane, such that

$$\|D(\mathbf{s}^*)\|_p = 0 \quad (87)$$

We may phase the assumptions $\lim_{t' \rightarrow \infty} (W_1 - W_0) = 0$ and $\lim_{t' \rightarrow \infty} \|D[t']\|_p = 0$ jointly as the condition

$$\lim_{t' \rightarrow \infty} \mathbf{s} = \mathbf{s}^* \quad (88)$$

By the Weierstrass definition of a limit, this is

$$\forall \varepsilon > 0, \exists t_0, \forall t > t_0, \quad \|\mathbf{s} - \mathbf{s}^*\|_p < \varepsilon. \quad (89)$$

For any ε , we have thus assumed that there exists some time t_0 beyond which \mathbf{s} is within ε of \mathbf{s}^* . In particular, we are free to choose ε small enough that the local dynamics of the system are well approximated by the linearization undertaken in the proof of Theorem 11:

Because the system is well approximated to first order within any sufficiently-small ε -neighborhood of the equilibrium hyperplane, the preimage of \mathbf{s}^* in the infinite-time limit within this neighborhood lies along the line through \mathbf{s}^* parallel to the *sole* eigenvector of the Jacobian with non-zero eigenvalue: \mathbf{v} .

When $\forall g, \delta(g, g+1) = 0$, \mathbf{v} is orthogonal to the internal equilibrium hyperplane (Eq. (21)), therefore, all states in the preimage of \mathbf{s}^* also satisfy $\forall g, \delta(g, g+1) = 0$ and exhibit zero p -total qualification rate disparity. We may then appeal to induction and Theorem 5 to note that the entire trajectory of the state must have had zero total disparity.

$$p \geq 1; \quad \lim_{t' \rightarrow \infty} (W_1 - W_0) = 0 \implies \left(\|D[t]\|_p = 0 \iff \lim_{t' \rightarrow \infty} \|D[t']\|_p = 0 \right) \quad (90)$$

This completes the proof.

A Series of Lemmas for Linear Stability Analysis

Lemma 11.1.

$$\left. \frac{\partial}{\partial \phi} \frac{W_1}{\overline{W}_g} \right|_{\text{eq}} = \frac{1}{W_{\text{eq}}} (1 - s_g) \left. \frac{\partial}{\partial \phi} (W_1 - W_0) \right|_{\text{eq}} \quad (91)$$

Proof of Lemma 11.1. We directly differentiate the expression evaluated at equilibrium, recalling that $W_{\text{eq}} = W_0|_{\text{eq}} = W_1|_{\text{eq}} = \overline{W}_g|_{\text{eq}} \forall g \in \mathcal{G}$ and $W_g = s_g W_1 + (1 - s_g) W_0$.

$$\left. \frac{\partial}{\partial \phi} \frac{W_1}{\overline{W}_g} \right|_{\text{eq}} = \left. \frac{1}{W_{\text{eq}}} \frac{\partial W_1}{\partial \phi} \right|_{\text{eq}} - \left. \frac{1}{W_{\text{eq}}} \frac{\partial \overline{W}_g}{\partial \phi} \right|_{\text{eq}} \quad (92a)$$

$$= \left. \frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} \left(W_1 - s_g W_1 - (1 - s_g) W_0 \right) \right|_{\text{eq}} \quad (92b)$$

$$= \left. \frac{1}{W_{\text{eq}}} (1 - s_g) \frac{\partial}{\partial \phi} (W_1 - W_0) \right|_{\text{eq}} \quad (92c)$$

Lemma 11.2. $\forall g \in \mathcal{G}$,

$$\frac{\partial \overline{s}}{\partial s_g} = \mu_g, \quad \frac{\partial \delta(g, g+1)}{\partial s_g} = 1, \quad \frac{\partial \delta(g-1, g)}{\partial s_g} = -1 \quad (93)$$

| *Proof of Lemma 11.2.* By Eq. (2) and Eq. (10), the result is immediate.

Lemma 11.3.

$$\frac{\partial s_g}{\partial \overline{s}} = 1 \quad (94)$$

| *Proof of Lemma 11.3.* By Eq. (12), the result is immediate.

Lemma 11.4.

$$\frac{\partial s_g}{\partial \delta(g, h)} = \begin{cases} 1 - \mu_1 - \mu_2 - \dots - \mu_g & h = g + 1 \\ \mu_1 + \mu_2 + \dots + \mu_{g-1} & h = g - 1 \end{cases} = \begin{cases} 1 + \beta_g & h = g + 1 \\ -\mu_g - \beta_g & h = g - 1 \end{cases} \quad (95)$$

$$\frac{\partial s_h}{\partial \delta(g, h)} = \begin{cases} -\mu_1 - \mu_2 - \dots - \mu_g & h = g + 1 \\ -1 + \mu_1 + \mu_2 + \dots + \mu_{g-1} & h = g - 1 \end{cases} = \begin{cases} \beta_g & h = g + 1 \\ -\mu_g - \beta_g - 1 & h = g - 1 \end{cases} \quad (96)$$

| *Proof of Lemma 11.4.* The result follows from Eq. (12), noting $\delta(g, h) = -\delta(h, g)$.

Lemma 11.5. Taking a partial derivative with respect to s_g while holding all other $s_h, h \neq g$ fixed,

$$\left. \frac{\partial}{\partial s_g[t]} s_g[t+1] \right|_{\text{eq}} = 1 + \mu_g \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \overline{s}} \right) s_g (1 - s_g) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (97)$$

Holding ϕ constant as well,

$$\left(\frac{\partial}{\partial s_g[t]} \right)_{\phi} s_g[t+1] \Big|_{\text{eq}} = 1 \quad (98)$$

When ϕ is held constant when taking a partial derivative with respect to s_g , we shall denote the partial derivative with ϕ in the subscript, as in the equation above, and omit this subscript otherwise.

| *Proof of Lemma 11.5.* Let us begin by proving the second equality, observing first that

$$\frac{\partial}{\partial s_g} \overline{W}_g = \frac{\partial}{\partial s_g} \left(s_g W_1 + (1 - s_g) W_0 \right) = W_1 - W_0 \quad (99)$$

With ϕ fixed, W_1 does not depend on s_g . Therefore, substituting $s_g[t+1] = s_g \frac{W_1}{W_g}$,

$$\left(\frac{\partial}{\partial s_g} \right)_{\phi} \left(s_g \frac{W_1}{W_g} \right) \Big|_{\text{eq}} = \frac{W_1}{W_g} \Big|_{\text{eq}} - s_g \frac{W_1}{W_g^2} (W_1 - W_0) \Big|_{\text{eq}} \quad (100a)$$

$$= \frac{W_{\text{eq}}}{W_{\text{eq}}} - 0 \quad (100b)$$

$$= 1 \quad (100c)$$

We next address the first equality. By Lemma 11.2,

$$\left(\frac{\partial \phi}{\partial s_g} \right) = \left(\frac{\partial \bar{s}}{\partial s_g} \right) \left(\frac{\partial \phi}{\partial \bar{s}} \right) = \mu_g \left(\frac{\partial \phi}{\partial \bar{s}} \right) \quad (101)$$

By Lemma 11.1,

$$\left(\frac{\partial}{\partial s_g} \right) \left(s_g \frac{W_1}{W_g} \right) \Big|_{\text{eq}} = \left(\frac{\partial}{\partial s_g} \right)_{\phi} \left(s_g \frac{W_1}{W_g} \right) \Big|_{\text{eq}} + s_g \left(\frac{\partial \phi}{\partial s_g} \right) \left(\frac{\partial}{\partial \phi} \frac{W_1}{W_g} \right) \Big|_{\text{eq}} \quad (102a)$$

$$= 1 + \mu_g \frac{s_g}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) (1 - s_g) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (102b)$$

Fact 11.1. We note when differentiating an expression \mathbf{g} with respect to an expression \mathbf{f} , each involving each s_g and ϕ (which depends on each s_g), we may invoke the chain rule to treat ϕ as an independent function input from the beginning, or we may treat the effect on ϕ due to perturbation of each s_g separately. It is for this reason that we have been explicit about which variables are fixed in the partial derivatives of Lemma 11.5.

$$\frac{\partial}{\partial s_g} \mathbf{f}(s_g, \phi) = \frac{\partial \phi}{\partial s_g} \frac{\partial \mathbf{f}}{\partial \phi} + \left(\frac{\partial \mathbf{f}}{\partial s_g} \right)_{\phi} \quad (103)$$

$$\frac{\partial \phi}{\partial \mathbf{f}} = \sum_{g \in \mathcal{G}} \frac{\partial s_g}{\partial \mathbf{f}} \frac{\partial \phi}{\partial s_g} \quad (104)$$

Therefore,

$$\frac{\partial}{\partial \mathbf{f}} \mathbf{g}(s_g, \phi) = \sum_{g \in \mathcal{G}} \left(\frac{\partial s_g}{\partial \mathbf{f}} \right) \frac{\partial}{\partial s_g} \mathbf{g}(s_g, \phi) \quad (105a)$$

$$= \sum_{g \in \mathcal{G}} \left(\frac{\partial s_g}{\partial \mathbf{f}} \right) \left(\frac{\partial \phi}{\partial s_g} \frac{\partial \mathbf{g}}{\partial \phi} + \left(\frac{\partial \mathbf{g}}{\partial s_g} \right)_{\phi} \right) \quad (105b)$$

$$= \sum_{g \in \mathcal{G}} \left(\frac{\partial s_g}{\partial \mathbf{f}} \frac{\partial \phi}{\partial s_g} \right) \frac{\partial \mathbf{g}}{\partial \phi} + \sum_{g \in \mathcal{G}} \left(\frac{\partial s_g}{\partial \mathbf{f}} \right) \left(\frac{\partial \mathbf{g}}{\partial s_g} \right)_{\phi} \quad (105c)$$

$$= \frac{\partial \phi}{\partial \mathbf{f}} \frac{\partial \mathbf{g}}{\partial \phi} + \sum_{g \in \mathcal{G}} \left(\frac{\partial s_g}{\partial \mathbf{f}} \right) \left(\frac{\partial \mathbf{g}}{\partial s_g} \right)_{\phi} \quad (105d)$$

For convenience, we will treat ϕ as an independent function input (i.e., we will invoke the chain rule as in Eq. (105d)) when proving Lemma 11.6, Lemma 11.7, and Lemma 11.8.

Lemma 11.6.

$$\frac{\partial(\bar{s}[t+1] - \bar{s}[t])}{\partial \bar{s}[t]} \Big|_{\text{eq}} = \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) \left(\sum_{g \in \mathcal{G}} \mu_g s_g (1 - s_g) \right) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (106)$$

Proof of Lemma 11.6. Noting that $\bar{s}[t+1]$ depends on each s_g and ϕ ,

$$\bar{s}[t+1] = \sum_{g \in \mathcal{G}} \mu_g s_g \frac{W_1(\phi)}{\bar{W}_g(\phi)} \quad (107)$$

we may use the chain rule (Fact 11.1),

$$\frac{\partial}{\partial \bar{s}} f(s_1, s_2, \dots, s_n, \phi) = \sum_{g \in \mathcal{G}} \frac{\partial s_g}{\partial \bar{s}} \left(\frac{\partial f}{\partial s_g} \right)_\phi + \frac{\partial \phi}{\partial \bar{s}} \frac{\partial f}{\partial \phi} \quad (108)$$

to compute, referencing Lemma 11.1, Lemma 11.2, Lemma 11.3, and Lemma 11.5,

$$\left. \frac{\partial(\bar{s}[t+1] - \bar{s}[t])}{\partial \bar{s}[t]} \right|_{\text{eq}} = \left. \frac{\partial \bar{s}[t+1]}{\partial \bar{s}[t]} - 1 \right|_{\text{eq}} \quad (109a)$$

$$= \left(\sum_{g \in \mathcal{G}} \left(\frac{\partial s_g}{\partial \bar{s}} \right) \left(\frac{\partial}{\partial s_g} \right)_\phi \left(\mu_g s_g \frac{W_1}{\bar{W}_g} \right) + \left(\frac{\partial \phi}{\partial \bar{s}} \right) \left(\frac{\partial}{\partial \phi} \sum_{g \in \mathcal{G}} \mu_g s_g \frac{W_1}{\bar{W}_g} \right) - 1 \right) \Big|_{\text{eq}} \quad (109b)$$

$$= \sum_{g \in \mathcal{G}} \mu_g + \left(\frac{\partial \phi}{\partial \bar{s}} \right) \frac{1}{W_{\text{eq}}} \sum_{g \in \mathcal{G}} \mu_g s_g (1 - s_g) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} - 1 \quad (109c)$$

$$= \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) \left(\sum_{g \in \mathcal{G}} \mu_g s_g (1 - s_g) \right) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (109d)$$

Lemma 11.7.

$$\left. \frac{\partial}{\partial \bar{s}[t]} \left(\delta(g, h)[t+1] - \delta(g, h)[t] \right) \right|_{\text{eq}} \quad (110a)$$

$$= \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) \delta(g, h) (1 - s_g - s_h) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (110b)$$

Proof of Lemma 11.7. Since \bar{s} and $\delta(g, h)$ are independent coordinates, the partial derivative of one with respect to the other at the same time is identically zero.

$$\frac{\partial}{\partial \bar{s}} \delta(g, h) = 0 \quad (111)$$

The left hand side of the target equality is therefore equal to

$$\left. \frac{\partial}{\partial \bar{s}[t]} \delta(g, h)[t+1] \right|_{\text{eq}}$$

From the chain rule (Fact 11.1), Lemma 11.1, Lemma 11.3, and Lemma 11.5, it follows that

$$\left. \frac{\partial}{\partial \bar{s}[t]} \delta(g, h)[t+1] \right|_{\text{eq}} \quad (112a)$$

$$= \left(\left(\frac{\partial \phi}{\partial \bar{s}} \right) \frac{\partial \delta(g, h)[t+1]}{\partial \phi} + \frac{\partial s_g}{\partial \bar{s}} \left(\frac{\partial s_g[t+1]}{\partial s_g[t]} \right)_{\phi} - \frac{\partial s_h}{\partial \bar{s}} \left(\frac{\partial s_h[t+1]}{\partial s_h[t]} \right)_{\phi} \right) \Big|_{\text{eq}} \quad (112b)$$

$$= \left(\frac{\partial \phi}{\partial \bar{s}} \right) \frac{\partial}{\partial \phi} \delta(g, h)[t+1] \Big|_{\text{eq}} \quad (112c)$$

$$= \left(\frac{\partial \phi}{\partial \bar{s}} \right) \frac{\partial}{\partial \phi} \left(s_g \frac{W_1}{W_g} - s_h \frac{W_1}{W_h} \right) \Big|_{\text{eq}} \quad (112d)$$

$$= \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) \left(s_g(1-s_g) - s_h(1-s_h) \right) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (112e)$$

$$= \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) \delta(g, h)(1-s_g-s_h) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (112f)$$

Lemma 11.8.

$$\left(\frac{\partial \bar{s}[t+1] - \bar{s}[t]}{\partial \delta(g, h)[t]} \right) \Big|_{\text{eq}} = 0 \quad (113)$$

$$\left(\frac{\partial \delta(g, h)[t+1] - \delta(g, h)[t]}{\partial \delta(g, h)[t]} \right) \Big|_{\text{eq}} = 0 \quad (114)$$

$$\left(\frac{\partial \delta(h, h+1)[t+1] - \delta(h, h+1)[t]}{\partial \delta(g, g+1)[t]} \right) \Big|_{\text{eq}} = 0, \quad \forall h \neq g \quad (115)$$

Proof of Lemma 11.8. By Theorem 1, ϕ depends only on \bar{s} , which is held constant during partial differentiation by $\delta(g, h)$. Therefore,

$$\frac{\partial \phi}{\partial \delta(g, h)} = 0 \quad (116)$$

Consider any expression f which depends *linearly* on each $s_g[t+1]$.

$$f(s_g : g \in \mathcal{G}) := \sum_{g \in \mathcal{G}} f_g s_g, \quad f_g \in \mathbf{R} \quad (117)$$

where we introduce a “vector builder” notation $\mathbf{s} = (s_g : g \in \mathcal{G})$ for brevity. We may use the linearity of differentiation to concisely deal with derivatives of linear combinations of f . We consider will expressions without explicit time dependence to correspond to time $[t]$. By the chain rule (Fact 11.1), Eq. (116), and Lemma 11.5,

$$\left. \frac{\partial}{\partial \delta(g, h)} (f[t+1] - f[t]) \right|_{\text{eq}} \quad (118a)$$

$$= \left(\sum_{i \in \mathcal{G}} \frac{\partial s_i}{\partial \delta(g, h)} \left(\frac{\partial}{\partial s_i} \right)_{\phi} + \frac{\partial \phi}{\partial \delta(g, h)} \frac{\partial}{\partial \phi} \right) f(s_i[t+1] - s_i[t] : i \in \mathcal{G}) \Big|_{\text{eq}} \quad (118b)$$

$$= f \left(\sum_{i \in \mathcal{G}} \frac{\partial s_i}{\partial \delta(g, h)} \left(\frac{\partial s_i[t+1]}{\partial s_i[t]} - \frac{\partial s_i[t]}{\partial s_i[t]} \right) : i \in \mathcal{G} \right) \quad (118c)$$

$$= f(0 : i \in \mathcal{G}) = 0 \quad (118d)$$

We conclude that perturbing any δ while holding \bar{s} constant has no effect on the evolution of dynamical variables that are linear in \mathbf{s} at equilibrium. This includes each δ and \bar{s} .

Proof of Theorem 11

Theorem 11 Statement. The Jacobian J simplifies to a scalar multiplied by a matrix with a single non-zero column \mathbf{v} in the last position.

$$J \Big|_{\text{eq}} = \frac{1}{W_{\text{eq}}} \left(\frac{d\phi}{d\bar{s}} \right) \left(\frac{d}{d\phi} (W_1 - W_0) \right) \left[\mathbf{0}^{(n \times n-1)} \Big| \mathbf{v} \right], \quad \mathbf{v} := \begin{bmatrix} \delta(1,2)(1-s_1-s_2) \\ \delta(2,3)(1-s_2-s_3) \\ \dots \\ \delta(n-1,n)(1-s_{n-1}-s_n) \\ \sum_{g \in \mathcal{G}} \mu_g s_g (1-s_g) \end{bmatrix} \quad (119)$$

Proof of Theorem 11. The zero entries in the Jacobian matrix are a consequence of Lemma 11.8. Lemma 11.7 and Lemma 11.6 provide us with the last column of the matrix J in the desired form:

$$\left. \frac{\partial \delta(g, h)[t+1] - \delta(g, h)[t]}{\partial \bar{s}[t]} \right|_{\text{eq}} = \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) \delta(g, h) (1-s_g - s_h) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (120)$$

$$\left. \frac{\partial (\bar{s}[t+1] - \bar{s}[t])}{\partial \bar{s}[t]} \right|_{\text{eq}} = \frac{1}{W_{\text{eq}}} \left(\frac{\partial \phi}{\partial \bar{s}} \right) \left(\sum_{g \in \mathcal{G}} \mu_g s_g (1-s_g) \right) \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \quad (121)$$

Proof of Corollary 11.1

Corollary 11.1 Statement.

At equilibrium, any state displacement vector with zero \bar{s} component is an eigenvector of J with eigenvalue 0, while \mathbf{v} is an eigenvector of J with eigenvalue λ :

$$\lambda := \left(\sum_{g \in \mathcal{G}} \mu_g s_g (1-s_g) \right) \frac{1}{W_{\text{eq}}} \left(\frac{d\phi}{d\bar{s}} \right) \left(\frac{d}{d\phi} (W_1 - W_0) \right) \Big|_{\text{eq}} \quad (122)$$

| *Proof of Corollary 11.1.* **Corollary 11.1** follows by inspection of J in Eq. (119).

Proof of Corollary 11.2

Corollary 11.2 Statement. As a consequence of Corollary 1.1, which states $\frac{d\phi}{d\bar{s}} < 0$, the eigenvalue λ in Eq. (22) is negative, (and the associated equilibrium hyperplane stable) iff $\frac{d}{d\phi} (W_1 - W_0) \Big|_{\text{eq}} > 0$. This prescribes precisely the value ϕ^+ for the stable equilibrium hyperplane.

| *Proof of Corollary 11.2.* This is a consequence of Corollary 1.1 and Corollary 11.1 given Assumption 1 (each s_g is interior) and the restriction of $W_y \in [0, \infty)$ as specified in the replicator equation (Eq. (7)). The eigenvalue λ is negative, (and the associated equilibrium hyperplane stable) iff

$$\left. \frac{d}{d\phi} (W_1 - W_0) \right|_{\text{eq}} > 0 \quad (123)$$

| This prescribes precisely the value ϕ^+ for the stable equilibrium hyperplane.

Proof of Theorem 13

Theorem 13 Statement. For policies defined by group-specific thresholds ϕ_g , the equivalence of these feature thresholds ($\forall g, \phi_g = \phi$) is necessary and sufficient to satisfy Equalized Odds given the group-independence of each q_y (Assumption 2).

| *Proof of Theorem 13.* The forward direction (group-independence satisfies Equalized Odds) follows from the group-independence of Q_y (Definition 2.1). The reverse direction follows from the same; specifically, as functions of ϕ ,

$$\Pr(\hat{Y} = 0 \mid Y = y) = Q_y(\phi) \quad (124)$$

$$\Pr(\hat{Y} = 1 \mid Y = y) = (1 - Q_y(\phi)) \quad (125)$$

| are each monotonic, and any specified value of $\Pr(\hat{Y} = \hat{y} \mid Y = y)$ corresponds to a unique ϕ value that must be shared by all groups.

Proof of Corollary 13.1

Corollary 13.1 Statement. Equalized Odds does not imply long-term fairness in our model.

Proof of Corollary 13.1. By contradiction, we have shown that a group-independent threshold policy satisfies Equalized Odds (Theorem 13), yet long-term fairness is violated by persistent qualification rate disparities (Theorem 10).

Proof of Theorem 14

Theorem 14 Statement. $\Theta(\epsilon)$ perturbations of a group-independent ϕ at internal equilibrium induce motion, which, to first-order approximation (*i.e.*, ignoring $\mathcal{O}(\epsilon^2)$ terms), is parallel to the eigenvector \mathbf{v} .

Proof of Theorem 14. We note that a perturbation to ϕ at internal equilibrium causes a change in state vector parallel to the eigenvector \mathbf{v} , where

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial \bar{s}} = \begin{bmatrix} \delta(1, 2)(1 - s_1 - s_2) \\ \delta(2, 3)(1 - s_2 - s_3) \\ \dots \\ \delta(n-1, n)(1 - s_{n-1} - s_n) \\ \sum_{g \in \mathcal{G}} \mu_g s_g (1 - s_g) \end{bmatrix} \quad (126)$$

By use of the chain rule with Lemma 11.7, or direct application of Lemma 11.1, we note

$$\left. \frac{\partial}{\partial \phi} \left(\delta(g, h)[t+1] - \delta(g, h)[t] \right) \right|_{\text{eq}} = \left(\frac{\partial \phi}{\partial \bar{s}} \right)^{-1} \left. \frac{\partial \delta(g, h)[t+1] - \delta(g, h)[t]}{\partial \bar{s}[t]} \right|_{\text{eq}} \quad (127a)$$

$$= \left. \frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0) \right|_{\text{eq}} \left(s_g(1 - s_g) - s_h(1 - s_h) \right) \Big|_{\text{eq}} \quad (127b)$$

$$= \left. \frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0) \right|_{\text{eq}} \delta(g, h)(1 - s_g - s_h) \quad (127c)$$

Likewise, pairing the chain rule with Lemma 11.6 or directly applying Lemma 11.1, we note

$$\left. \frac{\partial}{\partial \phi} \left(\bar{s}[t+1] - \bar{s}[t] \right) \right|_{\text{eq}} = \left(\frac{\partial \phi}{\partial \bar{s}} \right)^{-1} \left. \frac{\partial (\bar{s}[t+1] - \bar{s}[t])}{\partial \bar{s}[t]} \right|_{\text{eq}} \quad (128a)$$

$$= \left. \frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0) \right|_{\text{eq}} \sum_g \mu_g s_g (1 - s_g) \quad (128b)$$

Together, our observations imply

$$\left. \frac{\partial}{\partial \phi} (s[t+1] - s[t]) \right|_{\text{eq}} = \left. \frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0) \right|_{\text{eq}} \mathbf{v} \quad (129)$$

and perturbation of ϕ induces motion parallel to \mathbf{v} . For readers familiar with gradient descent but new to linear stability analysis, we offer the intuition that \mathbf{v} is parallel to the gradient of ϕ in state space.

Proof of Theorem 17

Theorem 17 Statement. Demographic parity requires sign-heterogeneous, group-dependent changes to the Laissez-Faire values of ϕ_g when π is non-trivial (does not uniformly accept (reject)).

Proof of Theorem 17. The policy adopted by a classifier subject to demographic parity is given by

$$\pi = \operatorname{argmax}_{\pi} \sum_{y, \hat{y}=0}^1 V_{y\hat{y}} \Pr_{\hat{Y}=\pi(X)} (Y = y, \hat{Y} = \hat{y}) \quad (130)$$

$$\text{subject to } \Pr(\hat{Y} = 1 \mid G = g) = \Pr(\hat{Y} = 1 \mid G = h) \quad \forall g, h \in \mathcal{G}$$

Without allowing group-dependent values of ϕ , the only solutions to π when groups have differing qualification rates are the trivial policies $\pi = 0$ and $\pi = 1$. We therefore consider a solution that permits group-dependent thresholds ϕ_g . We solve for these thresholds using the method of Lagrange multipliers. In the s_g state basis, this requires that we satisfy, for Lagrange multipliers $L_h \in (-\infty, \infty)$, $h \in \{1, 2, \dots, n-1\}$, the set of equations

$$\nabla_{\phi} u = \nabla_{\phi} \left(\sum_{h=1}^{n-1} L_h c_h \right) \quad (131)$$

where ∇_{ϕ} denotes the vector operator such that the g th component is the partial derivative with respect to ϕ_g ; u is the utility to be maximized; and each $c_h = 0$ represents a pairwise constraint between the probabilities of accepting an agent from two different groups (h and $h+1$).

$$u := \sum_{y, \hat{y}=0}^1 V_{y\hat{y}} \Pr_{\hat{Y}=\pi(X)} (Y = y, \hat{Y} = \hat{y}) \quad (132a)$$

$$= \sum_g \mu_g \begin{pmatrix} V_{0\hat{0}}(1-s_g)Q_0(\phi_g) \\ +V_{0\hat{1}}(1-s_g)(1-Q_0(\phi_g)) \\ +V_{1\hat{0}}s_gQ_1(\phi_g) \\ +V_{1\hat{1}}s_g(1-Q_1(\phi_g)) \end{pmatrix} \quad (132b)$$

$$c_h := \begin{pmatrix} s_h Q_1(\phi_h) + (1-s_h)Q_0(\phi_h) \\ -s_{h+1}Q_1(\phi_{h+1}) - (1-s_{h+1})Q_0(\phi_{h+1}) \end{pmatrix} \quad (133)$$

Defining $L_0 = L_n = 0$ for notational convenience, Eq. (131) simplifies to a set n equations indexed by $g \in \{1, 2, \dots, n\}$

$$\mu_g \left((V_{0\hat{0}} - V_{0\hat{1}})q_0(\phi_g)(1-s_g) + (V_{1\hat{0}} - V_{1\hat{1}})q_1(\phi_g)s_g \right) \quad (134a)$$

$$= (L_g - L_{g-1}) \left(s_g q_1(\phi_g) + (1-s_g)q_0(\phi_g) \right) \quad (134b)$$

From which the perturbed values ϕ_g may be derived:

$$\frac{q_1(\phi_g)}{q_0(\phi_g)} = \left(\frac{V_{0\hat{0}} - V_{0\hat{1}} - \gamma_g}{V_{1\hat{1}} - V_{1\hat{0}} + \gamma_g} \right) \left(\frac{1-s_g}{s_g} \right) \quad (135a)$$

$$\gamma_g := \frac{L_g - L_{g-1}}{\mu_g}; \quad L_g = \frac{\partial u}{\partial c_g} \quad (135b)$$

We compare this equation with Eq. (45), noting that when each $\gamma_g = 0$ (*i.e.*, requiring that constraints c_g are not active at locally optimal utility u), we recover a Laissez-fair policy:

$$\frac{q_1(\phi)}{q_0(\phi)} = \left(\frac{V_{0\hat{0}} - V_{0\hat{1}}}{V_{1\hat{1}} - V_{1\hat{0}}} \right) \left(\frac{1-s_g}{s_g} \right) \quad (136)$$

For interpretation of the Lagrange multipliers L_g , also known as the *dual variables*, we refer the reader to Boyd et al. [43]. By the monotonicity of q_1/q_0 , the effect of γ_g in determining ϕ_g is therefore a perturbation to ϕ , the sign of which is inverted relative to the sign of γ_g . Finally, having defined $L_0 = L_n = 0$, as a telescoping sum,

$$\sum_{g=1}^n (L_g - L_{g-1}) = 0 \quad (137)$$

Therefore,

$$\sum_{g=1}^n \mu_g \gamma_g = 0 \quad (138)$$

This guarantees in turn that set of group-specific *changes* to the group-specific values ϕ_g defined by a Laissez-Fair policy (Eq. (136)) must be sign-heterogeneous to satisfy Demographic Parity.

We comment that a solution for each γ_g that satisfying constraints c_g requires the solution of differential equation(s) in q_y (i.e., equation(s) involving both $q_y(\phi_g)$ and $Q_y(\phi_g)$ simultaneously). This most apparent if we appeal to the chain rule to write

$$L_g = \frac{\partial u}{\partial c_g} = \frac{\partial s_g}{\partial c_g} \frac{\partial u}{\partial s_g} + \frac{\partial s_{g+1}}{\partial c_g} \frac{\partial u}{\partial s_{g+1}} \quad (139a)$$

$$+ \frac{\partial Q_0(\phi_g)}{\partial c_g} \frac{\partial u}{\partial Q_0(\phi_g)} + \frac{\partial Q_1(\phi_g)}{\partial c_g} \frac{\partial u}{\partial Q_1(\phi_g)} \quad (139b)$$

$$+ \frac{\partial Q_0(\phi_{g+1})}{\partial c_g} \frac{\partial u}{\partial Q_0(\phi_{g+1})} + \frac{\partial Q_1(\phi_{g+1})}{\partial c_g} \frac{\partial u}{\partial Q_1(\phi_{g+1})} \quad (139c)$$

which, after some simplification, yields an expression in terms of Q_y :

$$L_g - L_{g-1} = \quad (140a)$$

$$\left(Q_1(\phi_{g+1}) - Q_0(\phi_{g+1}) \right) \left(Q_0(\phi_{g+1}) (V_{0i} - V_{00}) + Q_1(\phi_{g+1}) (V_{10} - V_{1i}) \right) \quad (140b)$$

$$- \left(Q_1(\phi_{g-1}) - Q_0(\phi_{g-1}) \right) \left(Q_0(\phi_{g-1}) (V_{0i} - V_{00}) + Q_1(\phi_{g-1}) (V_{10} - V_{1i}) \right) \quad (140c)$$

Considering that we treat arbitrary q_y subject to Assumption 3, analytically solving an equation in q and Q simultaneously is not practical for our purposes.

Proof of Theorem 18

Theorem 18 Statement. On the stable internal equilibrium hyperplane, infinitesimal perturbation of Φ by

$$\Delta_g \Phi := -\epsilon \delta(g, g+1) \left(\frac{\alpha_g}{s_1(1-s_1)}, \dots, \frac{\alpha_g}{s_g(1-s_g)}, \frac{\beta_g}{s_{g+1}(1-s_{g+1})}, \dots, \frac{\beta_g}{s_n(1-s_n)} \right) \quad (141a)$$

$$\alpha_g := (\mu_{g+1} + \mu_{g+2} + \dots + \mu_n), \quad \beta_g := -(\mu_1 + \mu_2 + \dots + \mu_g) \quad (141b)$$

will induce motion in the system preserving \bar{s} and each $\delta(h, h+1)$ for $h \neq g$. The value of $\delta(g, g+1)$ will be diminished by a ratio proportional to the **strength parameter** $\epsilon > 0$.

Proof of Theorem 18. For convenience, on an equilibrium hyperplane, we will write as equivalent statements

$$\frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} = \frac{\partial}{\partial \phi_g} (W_1^g - W_0^g) \Big|_{\text{eq}} \quad (142)$$

We first generalize Lemma 11.1 for group-dependent feature thresholds ϕ_g , each perturbed from $\phi_g = \phi$ at equilibrium and but applied only to the corresponding group g .

$$\frac{\partial}{\partial \phi_g} \left(s_h[t+1] - s_h[t] \right) \Big|_{\text{eq}} = \frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \begin{cases} s_g(1-s_g) & g = h \\ 0 & h \neq g \end{cases} \quad (143)$$

It follows from the definition of \bar{s} that

$$\frac{\partial}{\partial \phi_g} \left(\bar{s}[t+1] - \bar{s}[t] \right) \Big|_{\text{eq}} = \frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \mu_g s_g (1-s_g) \quad (144)$$

and, by the definition of $\delta(h, h+1)$,

$$\frac{\partial}{\partial \phi_g} \left(\delta(h, h+1)[t+1] - \delta(h, h+1)[t] \right) \Big|_{\text{eq}} = \quad (145a)$$

$$\left(\frac{1}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0) \Big|_{\text{eq}} \right) \begin{cases} s_g(1-s_g) & g = h \\ -s_g(1-s_g) & g = h+1 \\ 0 & \text{otherwise} \end{cases} \quad (145b)$$

We may now prove that perturbation of the vector Φ by the vector $\Delta_g \Phi = (\Delta_g \phi_1, \Delta_g \phi_2, \dots, \Delta_g \phi_n)$ causes the system to maintain its current value of \bar{s} . We sum the contribution due to each $\Delta_g \phi_h$,

where $\langle \cdot, \cdot \rangle$ denotes the inner product, ∇_{Φ} denotes a gradient taken with respect to the components of Φ , and by linearity, $\langle \Delta_g \Phi, \nabla_{\Phi} \rangle$ is an operator that perturbs the system with change $\Delta_g \Phi$. Linear proportionality is denoted with \propto .

$$\Delta_g(\bar{s}[t+1] - \bar{s}[t])\Big|_{\text{eq}} = \langle \Delta_g \Phi, \nabla_{\Phi} \rangle (\bar{s}[t+1] - \bar{s}[t])\Big|_{\text{eq}} \quad (146a)$$

$$= \sum_{h=1}^n (\Delta_g \phi_h) \frac{\partial}{\partial \phi_h} (\bar{s}[t+1] - \bar{s}[t])\Big|_{\text{eq}} \quad (146b)$$

$$= \frac{-\epsilon \delta(g, g+1)}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0)\Big|_{\text{eq}} \left(\sum_{h=1}^g \mu_h \alpha_g + \sum_{h=g+1}^n \mu_h \beta_g \right) \quad (146c)$$

$$\propto (-\beta_g \alpha_g + \alpha_g \beta_g) \quad (146d)$$

$$= 0 \quad (146e)$$

Next, by Eq. (145a), we consider the effect that the perturbation $\Delta_g \Phi$ has on each $\delta(h, h+1)$ at equilibrium.

$$\Delta_g(\delta(h, h+1)[t+1] - \delta(h, h+1)[t])\Big|_{\text{eq}} \quad (147a)$$

$$= \langle \Delta_g \Phi, \nabla_{\Phi} \rangle (\delta(h, h+1)[t+1] - \delta(h, h+1)[t])\Big|_{\text{eq}} \quad (147b)$$

$$= \sum_{i=1}^n (\Delta_g \phi_i) \frac{\partial}{\partial \phi_i} (\delta(h, h+1)[t+1] - \delta(h, h+1)[t])\Big|_{\text{eq}} \quad (147c)$$

$$= \left(\frac{-\epsilon \delta(g, g+1)}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0)\Big|_{\text{eq}} \right) \begin{cases} \alpha_g & h \leq g \\ \beta_g & h > g \end{cases} \quad (147d)$$

$$- \left(\frac{-\epsilon \delta(g, g+1)}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0)\Big|_{\text{eq}} \right) \begin{cases} \alpha_g & h+1 \leq g \\ \beta_g & h+1 > g \end{cases} \quad (147e)$$

$$= \left(\frac{-\epsilon \delta(g, g+1)}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0)\Big|_{\text{eq}} \right) \begin{cases} \alpha_g - \beta_g & g = h \\ 0 & g \neq h \end{cases} \quad (147f)$$

Since $\alpha_g - \beta_g = 1$ by Eq. (1), We see that the discrete velocity in $\delta(g, g+1)$ induced by $\Delta_g \Phi$ is

$$-\epsilon \frac{\delta(g, g+1)}{W_{\text{eq}}} \frac{\partial}{\partial \phi} (W_1 - W_0)\Big|_{\text{eq}} \quad (148)$$

On the stable equilibrium hyperplane, where $\frac{\partial}{\partial \phi} (W_1 - W_0) > 0$ and initial discrete velocity in $\delta(g, g+1)$ is zero, the prescribed perturbation proportionately *opposes* $\delta(g, g+1)$.

C Additional Figures

For all settings, we display the simulated dynamics for two groups, subject to different global interventions. Streamlines approximate system time evolution. q_0 and q_1 are Gaussians with unit variance and have means -1 and 1 , respectively. The figures included herein are provided with little analysis and are intended to prompt further consideration for the curious reader.

C.1 Additional Variables of Interest

In addition to the acceptance rate for Group 1 (blue; first row), we plot the false positive rate for Group 1 (red; second row) and the false negative rate for Group 1 (green; third row).

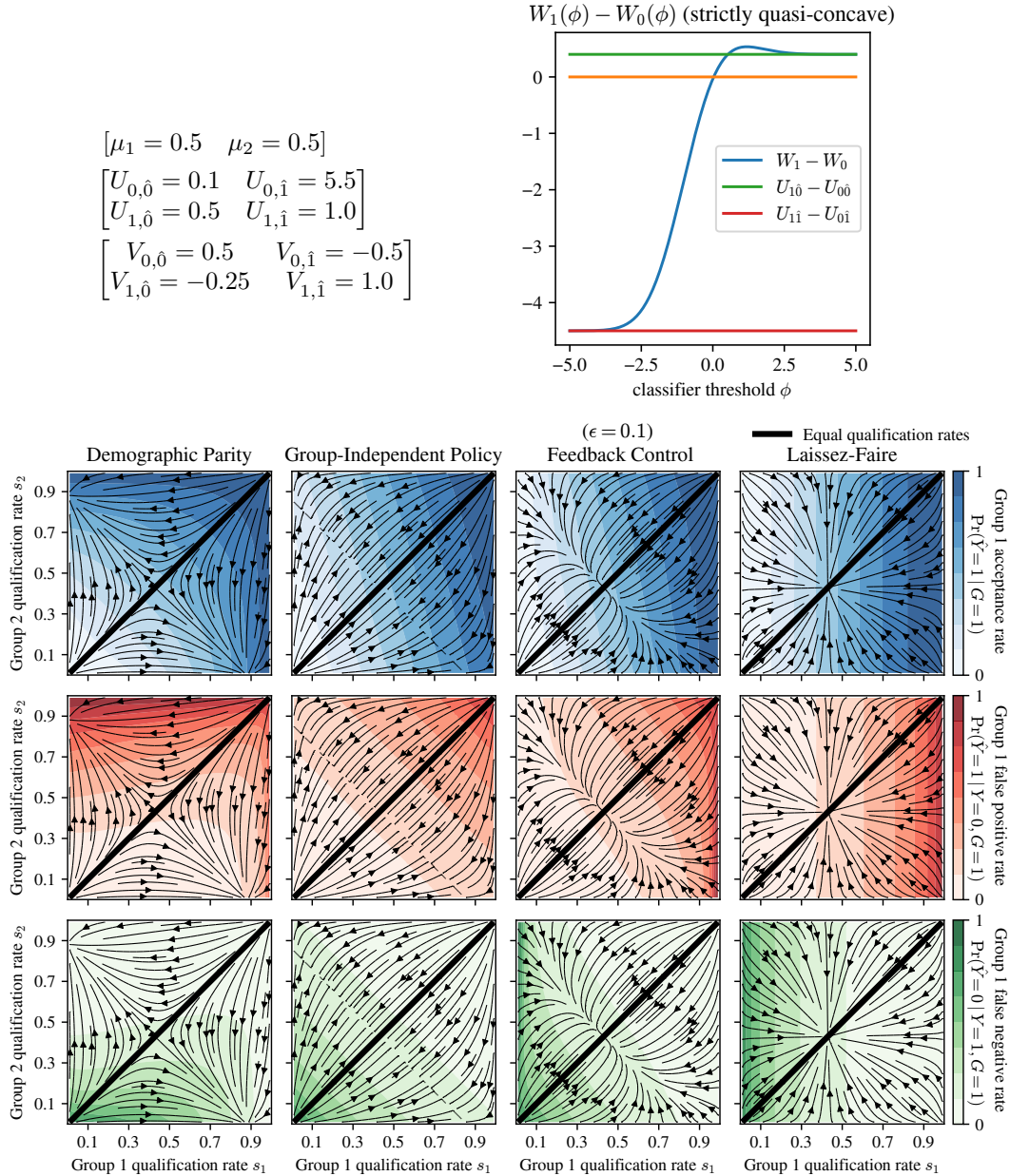


Figure 4: **Setting 1** (Analyzed in Section 4)

C.2 Different U and V

Classifier decisions will differ for a given state s when V is modified. Similarly, the success of different strategies update with different U values. The qualitative behavior of the system ultimately depends on the shape of $W_1 - W_0$ as a function of ϕ .

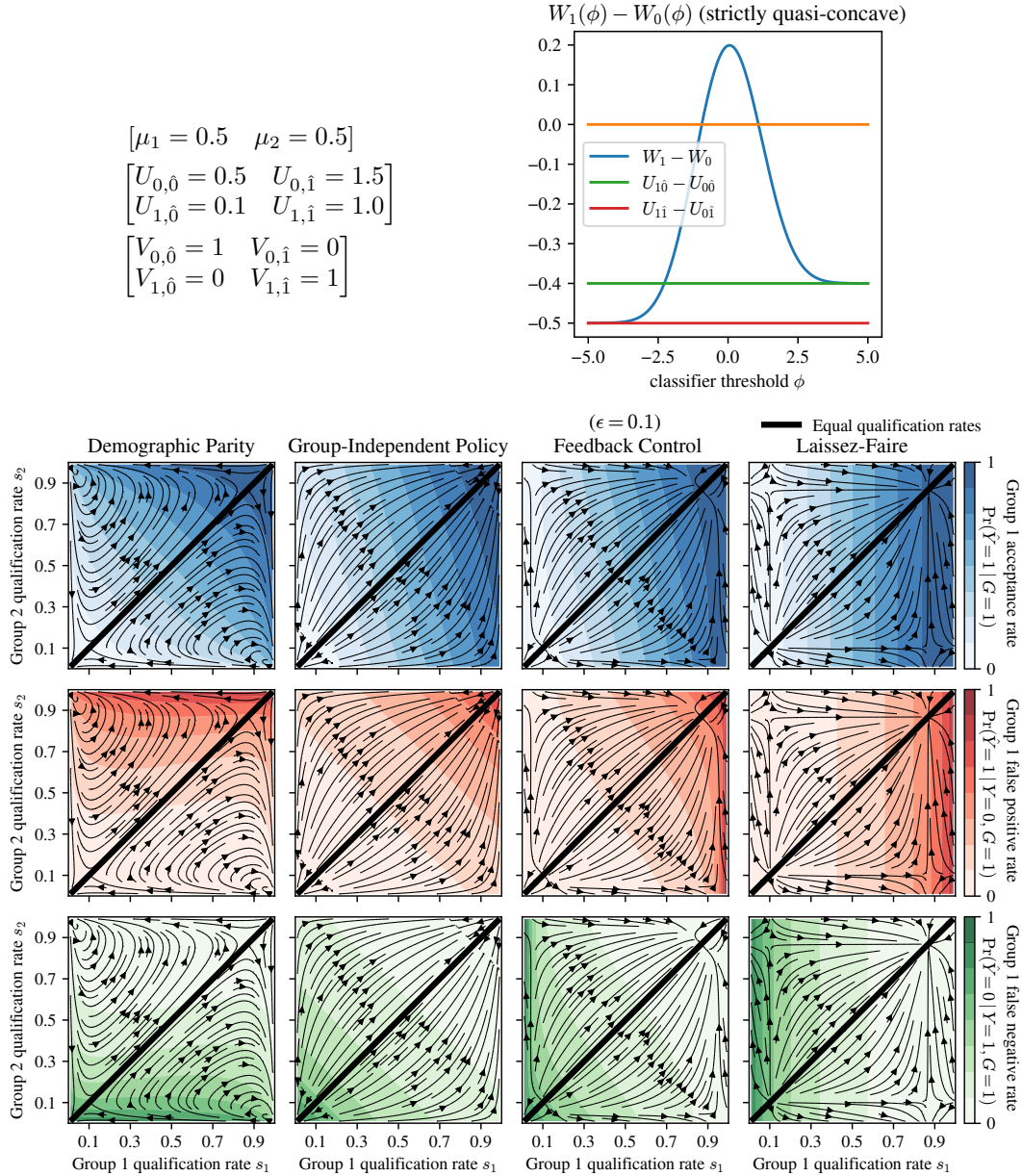


Figure 5: **Setting 2** (Stable *and* unstable hyperplanes)

$$\begin{aligned} & [\mu_1 = 0.5 \quad \mu_2 = 0.5] \\ & \begin{bmatrix} U_{0,0} = 0.5 & U_{0,1} = 0.5 \\ U_{1,0} = 0.1 & U_{1,1} = 1.5 \end{bmatrix} \\ & \begin{bmatrix} V_{0,0} = 10.0 & V_{0,1} = 0.0 \\ V_{1,0} = 1.0 & V_{1,1} = 1.5 \end{bmatrix} \end{aligned}$$

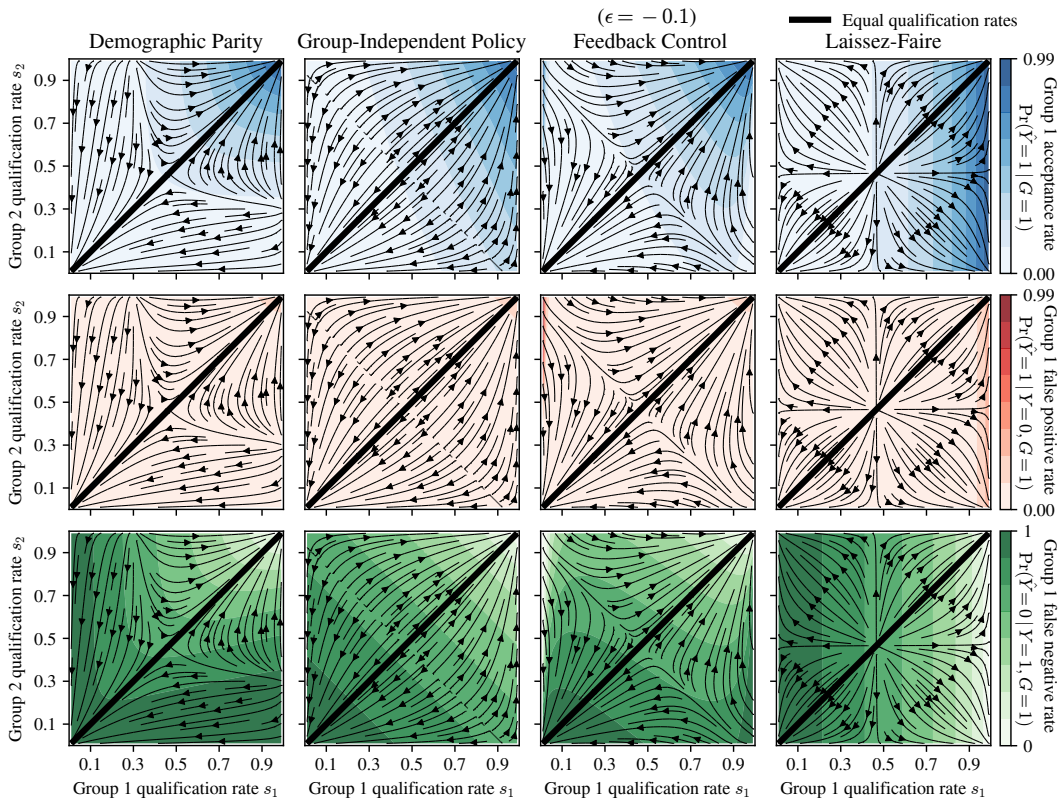
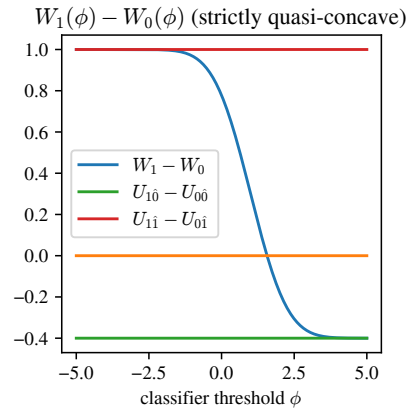
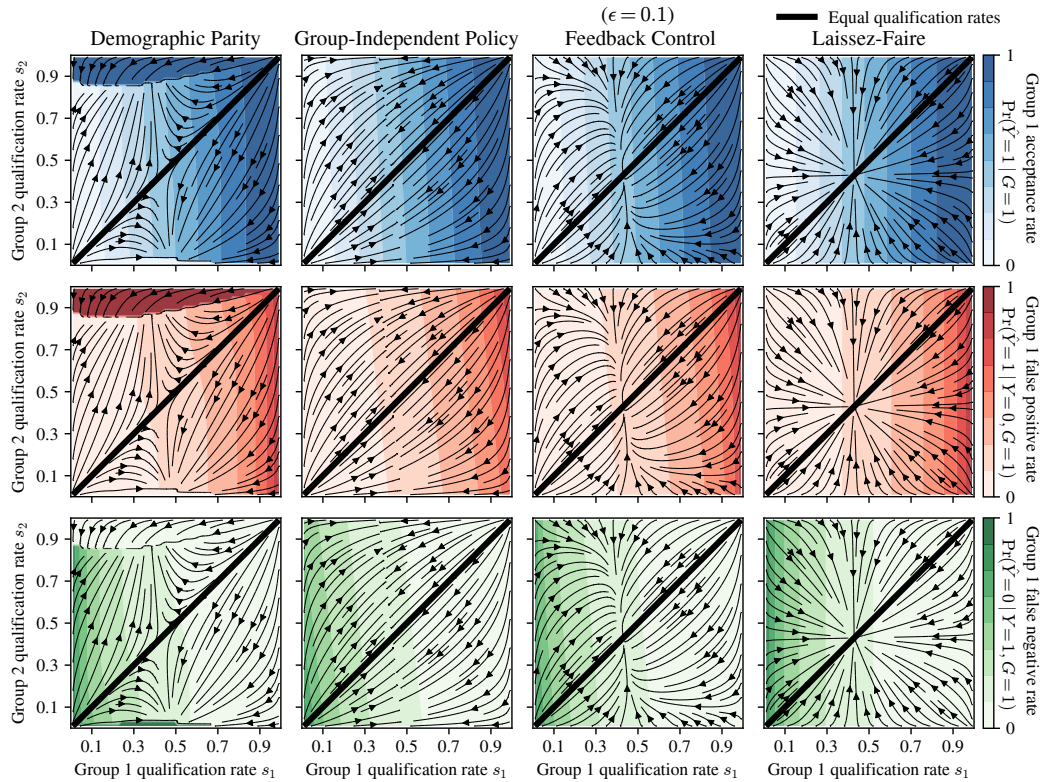
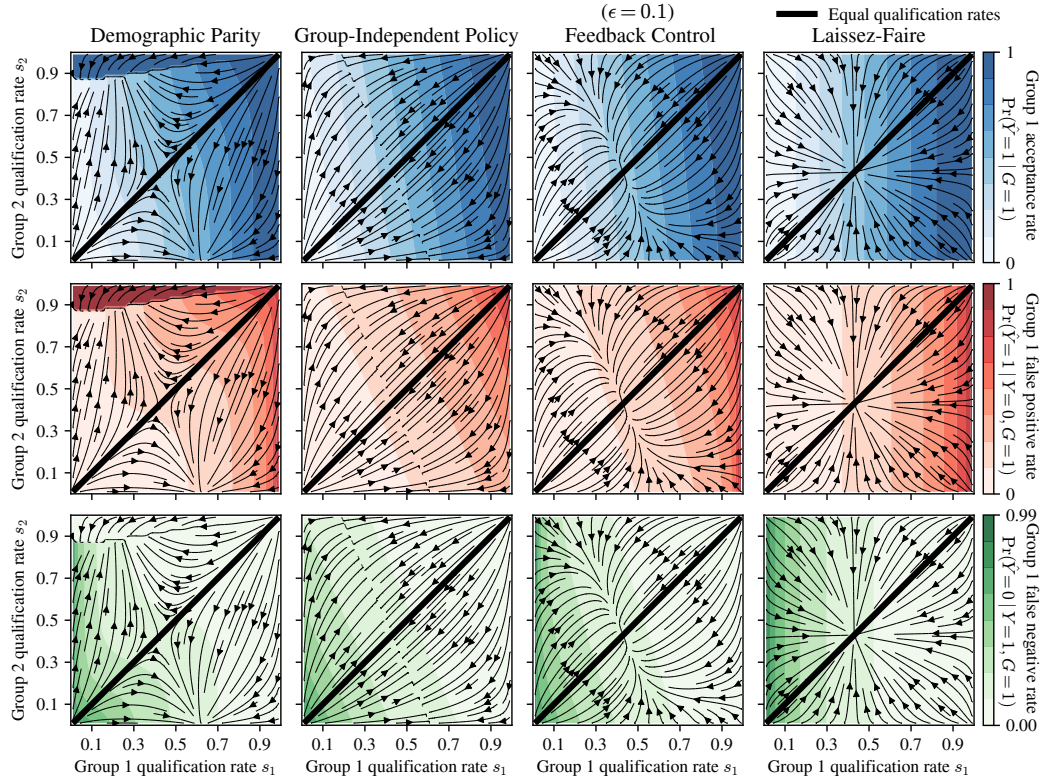
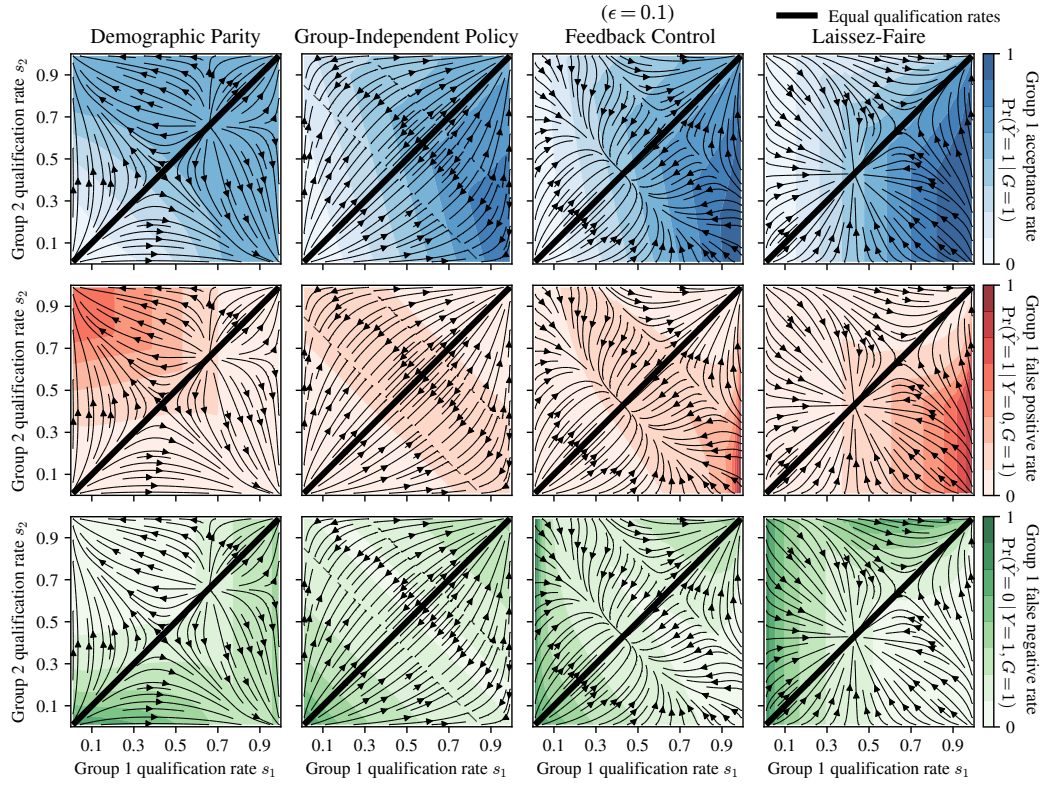


Figure 6: **Setting 3** (Only an unstable hyperplane; Note the negative value of ϵ for Feedback Control.)

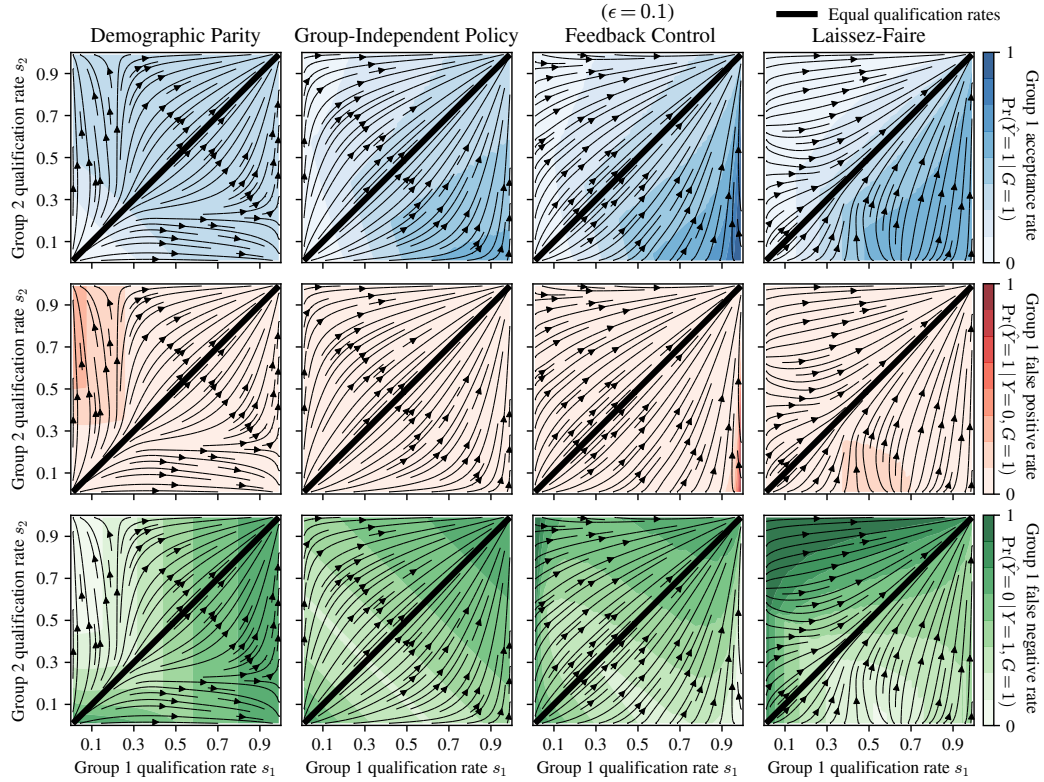
C.3 Different Group Sizes



C.4 Limited Space for Acceptance



(8a) Setting 1, but the classifier is limited to accepting $\Pr(\hat{Y}) < 0.6$



(8b) Setting 1, in the classifier is limited to accepting $\Pr(\hat{Y}) < 0.3$

C.5 Other Models

For completeness, we picture the dynamics of other models. Specifically, a Markov model like that of Zhang et al. [23] (Fig. 9) and the “best response” model of Coate and Loury [21] (Fig. 10). Note that the setting of Coate and Loury [21] assumes that agents privately know their own costs for becoming qualified, which are sampled from a group-independent distribution, rather than being uniform for all agents. We use the following set of parameters.

$$\begin{aligned} & [\mu_1 = 0.5 \quad \mu_2 = 0.5] \\ & \begin{bmatrix} T_{0,\hat{0}} = 0.2 & T_{0,\hat{1}} = 0.5 \\ T_{1,\hat{0}} = 0.1 & T_{1,\hat{1}} = 0.8 \end{bmatrix} \\ & \begin{bmatrix} V_{0,\hat{0}} = 0.0 & V_{0,\hat{1}} = -1.0 \\ V_{1,\hat{0}} = 0.0 & V_{1,\hat{1}} = 1.3 \end{bmatrix} \end{aligned}$$

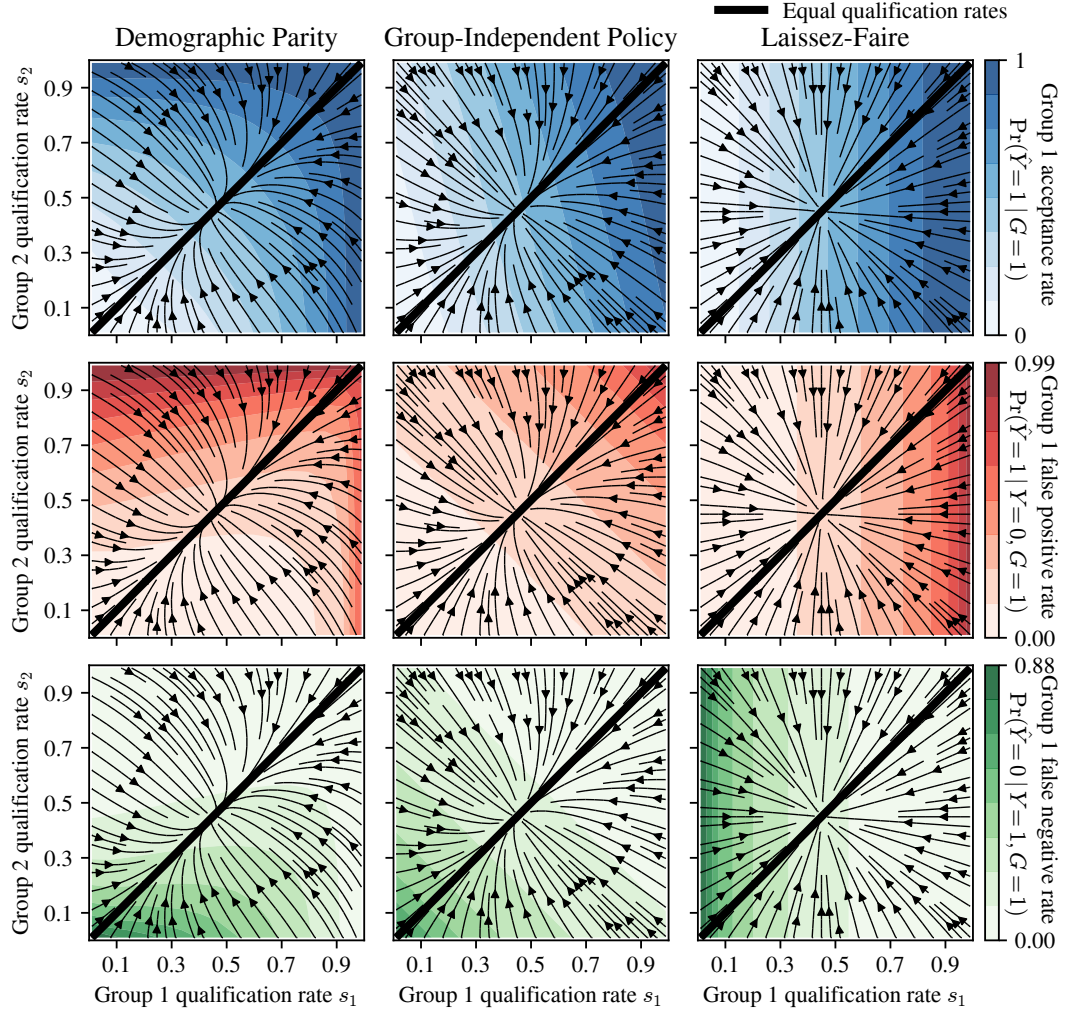


Figure 9: The classifier of Setting 1, but the dynamics of Zhang et al. [23], where the probability of an agent becoming qualified in the next round given outcome y, \hat{y} , denoted $T_{y,\hat{y}}$, given as above. We assume T is group-independent; under this assumption, disparity in qualification rates cannot persist.

$$\begin{bmatrix} \mu_1 = 0.5 & \mu_2 = 0.5 \\ V_{0,\hat{0}} = 0.0 & V_{0,\hat{1}} = -500.0 \\ V_{1,\hat{0}} = 0.0 & V_{1,\hat{1}} = 1.0 \end{bmatrix}$$

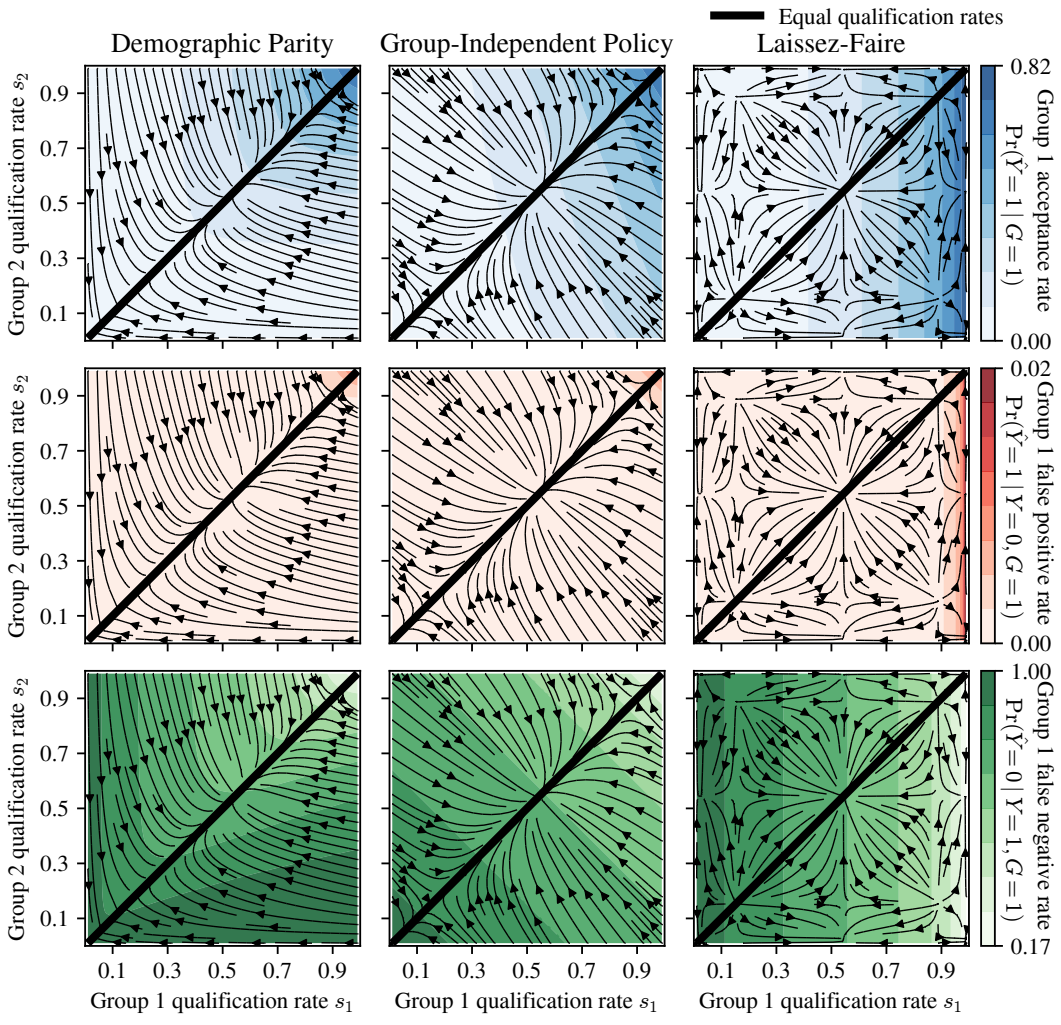
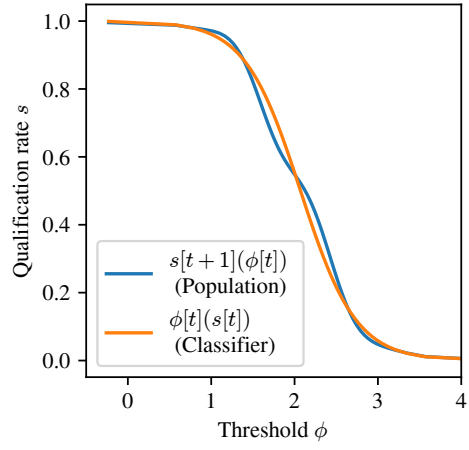


Figure 10: The classifier of Setting 1, but the population response model of Coate and Loury [21]. The intersections of the curves shown above the phase portraits correspond to the possible fixed points of the system in qualification rate; these intersections had to be manufactured with a distribution of costs, known to agents privately, associated with qualification.