# Discovering conflicting groups in signed networks
# Supplementary material

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## A Proof of Lemma 1

**Proof:** We have $\|\mathbf{v}\|_2 = 1$ and, without loss of generality, we can assume that the coordinates of $\mathbf{v}$ are sorted in non-increasing order. Let $\mathcal{T} = \{t_i\}_{i=0}^{n+1}$ be all possible thresholds for $\mathbf{v}$ and $\mathcal{T}' = \{t'_i\}_{i=0}^{n+1}$ be all possible thresholds for $-\mathbf{v}$. Recall the definition of $\theta(\cdot,\cdot)$ from Section 3 that $\theta(\mathbf{a},\mathbf{b}) = \arccos(\langle \mathbf{a},\mathbf{b}\rangle / \|\mathbf{a}\|_2\|\mathbf{b}\|_2) \in [0,\pi]$ for any two nonnegative vectors $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$, so $\sin\theta(\mathbf{a},\mathbf{b})$ is always non-negative. Let $\mathbf{u}^*$ be the minimizer of $\sin\theta(\mathbf{v},\mathbf{u})$ over all $\mathbf{u} \in \Gamma(\mathbf{v}) \cup \Gamma(-\mathbf{v})$.

For simplicity, we assume $\mathbf{u}^* \in \Gamma(\mathbf{v})$ and $\langle \mathbf{v},\mathbf{u}^*\rangle \geq 0$. This is because if the dot product is negative, we can make it positive by reversing the sign of $\mathbf{v}$. Let $k_1^*, k_2^*$ be the two thresholds such that $\mathbf{u}^* = \sigma_{t_{k_1^*}, t_{k_2^*}}(\mathbf{v})$. We will show that $\sin\theta(\mathbf{v},\mathbf{u}) \geq \sin\theta(\mathbf{v},\mathbf{u}^*)$ for any $\mathbf{u} \in \{0,-1,q\}^n$.

Fix any $\mathbf{u} \in \{0,-1,q\}^n$. Our first step is to identify the coordinates that $\mathbf{u}_i \neq \mathbf{u}_i^*$, denoted by $\mathcal{I} = \{j : \mathbf{u}_j \neq \mathbf{u}_j^*\}$. Moreover, since $\mathbf{u}_j^* = q$ for all $j \leq k_1^*$, $\mathbf{u}_j^* = -1$ for all $j \geq k_2^*$, and $\mathbf{u}_j^* = 0$ for all $j \in (k_1^*, k_2^*)$, we further divide $\mathcal{I}$ into 6 disjoint subsets:

$$\mathcal{I}_{11} = \{j \in \mathcal{I} : \mathbf{u}_j = 0, j \leq k_1^*\}, \qquad \mathcal{I}_{12} = \{j \in \mathcal{I} : \mathbf{u}_j = -1, j \leq k_1^*\},$$
$$\mathcal{I}_{21} = \{j \in \mathcal{I} : \mathbf{u}_j = q, j \in (k_1^*, k_2^*)\}, \qquad \mathcal{I}_{22} = \{j \in \mathcal{I} : \mathbf{u}_j = -1, j \in (k_1^*, k_2^*)\},$$
$$\mathcal{I}_{31} = \{j \in \mathcal{I} : \mathbf{u}_j = 0, j \geq k_2^*\}, \qquad \mathcal{I}_{32} = \{j \in \mathcal{I} : \mathbf{u}_j = q, j \geq k_2^*\}.$$

Denote the overall division by $k_1^*$ and $k_2^*$ by $\mathcal{I}_1 = \mathcal{I}_{11} \cup \mathcal{I}_{12}$, $\mathcal{I}_2 = \mathcal{I}_{21} \cup \mathcal{I}_{22}$, and $\mathcal{I}_3 = \mathcal{I}_{31} \cup \mathcal{I}_{32}$.

We claim that for any such $\mathbf{u}$, there exists a vector $\tilde{\mathbf{u}} \in \Gamma(\mathbf{v}) \cup \Gamma(-\mathbf{v})$ such that $\sin\theta(\mathbf{v},\mathbf{u}) \geq \sin\theta(\mathbf{v},\tilde{\mathbf{u}})$, which is sufficient to complete the proof since $\mathbf{u}^*$ is the minimizer of $\sin\theta(\mathbf{v},\mathbf{u})$ for all $\mathbf{u} \in \Gamma(\mathbf{v}) \cup \Gamma(-\mathbf{v})$. We will show how to find such vector $\tilde{\mathbf{u}}$ by examining the following two cases:

(Case 1) $\langle \mathbf{v},\mathbf{u}\rangle \geq 0$:

Let $c_1 = |\mathcal{I}_{21}| - |\mathcal{I}_1|$ and $c_2 = |\mathcal{I}_{22}| - |\mathcal{I}_3|$. The claim is proved by setting $\tilde{\mathbf{u}} = \sigma_{t_{k_1^*+c_1}, t_{k_2^*-c_2}}(\mathbf{v})$, which is justified by the following two observations.

First, observe that $\|\tilde{\mathbf{u}}\|_2 \leq \|\mathbf{u}\|_2$ because $\|\tilde{\mathbf{u}}\|_2^2 + |\mathcal{I}_{12}| + q^2|\mathcal{I}_{32}| = \|\mathbf{u}\|_2^2$.

Second, write $\langle \mathbf{v},\mathbf{u}\rangle$ as

$$\langle \mathbf{v},\mathbf{u}\rangle = \langle \mathbf{v},\mathbf{u}^*\rangle + q\left(-\sum_{j\in\mathcal{I}_1}\mathbf{v}_j + \sum_{j\in\mathcal{I}_{21}\cup\mathcal{I}_{32}}\mathbf{v}_j\right) + \left(\sum_{j\in\mathcal{I}_3}\mathbf{v}_j + \sum_{j\in\mathcal{I}_{12}\cup\mathcal{I}_{22}}(-\mathbf{v}_j)\right). \quad (1)$$

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Notice that some terms of the summation in Equation (1) are negative, in particular,

$$\sum_{j\in\mathcal{I}_{12}}(-\mathbf{v}_j)+\sum_{j\in\mathcal{I}_{32}}q\mathbf{v}_j<0,$$

since $\mathbf{v}_j>0$ for all $j\in\mathcal{I}_{12}$, and $\mathbf{v}_j<0$ for all $j\in\mathcal{I}_{32}$.

Therefore, we have

$$\langle\mathbf{v},\mathbf{u}\rangle\leq\langle\mathbf{v},\mathbf{u}^*\rangle+q\left(-\sum_{j\in\mathcal{I}_1}\mathbf{v}_j+\sum_{j\in\mathcal{I}_{21}}\mathbf{v}_j\right)+\left(\sum_{j\in\mathcal{I}_3}\mathbf{v}_j+\sum_{j\in\mathcal{I}_{22}}(-\mathbf{v}_j)\right). \tag{2}$$

Since $\mathbf{v}$ is sorted non-increasingly, the latter two terms in (2) are smaller than

$$q\left(-\sum_{j=1}^{|\mathcal{I}_1|}\mathbf{v}_{k_1^*-j}+\sum_{j=1}^{|\mathcal{I}_{21}|}\mathbf{v}_{k_1^*-|\mathcal{I}_1|+j}\right)+\left(\sum_{j=1}^{|\mathcal{I}_3|}\mathbf{v}_{k_2^*+j}+\sum_{j=1}^{|\mathcal{I}_{22}|}(-\mathbf{v}_{k_2^*+|\mathcal{I}_3|-j})\right).$$

That is,

$$\begin{aligned}\langle\mathbf{v},\mathbf{u}\rangle&\leq\langle\mathbf{v},\mathbf{u}^*\rangle+q\left(-\sum_{j=1}^{|\mathcal{I}_1|}\mathbf{v}_{k_1^*-j}+\sum_{j=1}^{|\mathcal{I}_{21}|}\mathbf{v}_{k_1^*-|\mathcal{I}_1|+j}\right)+\left(\sum_{j=1}^{|\mathcal{I}_3|}\mathbf{v}_{k_2^*+j}+\sum_{j=1}^{|\mathcal{I}_{22}|}(-\mathbf{v}_{k_2^*+|\mathcal{I}_3|-j})\right)\\&=\langle\mathbf{v},\tilde{\mathbf{u}}\rangle\end{aligned}$$

Hence, we have $0\leq\cos\theta(\mathbf{v},\mathbf{u})\leq\cos\theta(\mathbf{v},\tilde{\mathbf{u}})$, which is equivalent to $\sin\theta(\mathbf{v},\mathbf{u})\geq\sin\theta(\mathbf{v},\tilde{\mathbf{u}})$ due to the non-negativity of $\sin\theta(\cdot,\cdot)$.

(Case 2) $\langle\mathbf{v},\mathbf{u}\rangle<0$:

Let $c_1=|\{j\in\mathcal{I}_{21}:\mathbf{v}_j<0\}|+|\mathcal{I}_{32}|$ and $c_2=|\{j\in\mathcal{I}_{22}:\mathbf{v}_j>0\}|+|\mathcal{I}_{12}|$. The claim is proved by setting $\tilde{\mathbf{u}}=\sigma_{t'_{c_1},t'_{c_2}}(-\mathbf{v})$, which is justified in the below two observations.

First, observe that $\|\tilde{\mathbf{u}}\|_2\leq\|\mathbf{u}\|_2$ because

$$\|\tilde{\mathbf{u}}\|_2^2+q^2|\{j\in\mathcal{I}_{21}:\mathbf{v}_j\geq0\}|+|\{j\in\mathcal{I}_{22}:\mathbf{v}_j\leq0\}|=\|\mathbf{u}\|_2^2.$$

Second, write $\langle\mathbf{v},\mathbf{u}\rangle$ by Equation (1) as

$$\langle\mathbf{v},\mathbf{u}\rangle=\langle\mathbf{v},\mathbf{u}^*\rangle+q\left(-\sum_{j\in\mathcal{I}_1}\mathbf{v}_j+\sum_{j\in\mathcal{I}_{21}\cup\mathcal{I}_{32}}\mathbf{v}_j\right)+\left(\sum_{j\in\mathcal{I}_3}\mathbf{v}_j+\sum_{j\in\mathcal{I}_{12}\cup\mathcal{I}_{22}}(-\mathbf{v}_j)\right). \tag{3}$$

Notice that some terms of the summation in Equation (3) are non-negative, in particular

$$\sum_{j\in\mathcal{I}_{21},\mathbf{v}_j\geq0}q\mathbf{v}_j+\sum_{j\in\mathcal{I}_{22},\mathbf{v}_j\leq0}(-\mathbf{v}_j)\geq0.$$

Therefore, by letting $\mathcal{I}_{21}^-=\{i\in\mathcal{I}_{21},\mathbf{v}_i<0\}$ and $\mathcal{I}_{22}^+=\{i\in\mathcal{I}_{22},\mathbf{v}_i>0\}$, we have

$$\begin{aligned}\langle\mathbf{v},\mathbf{u}\rangle&\geq\langle\mathbf{v},\mathbf{u}^*\rangle+q\left(-\sum_{j\in\mathcal{I}_1}\mathbf{v}_j+\sum_{j\in\mathcal{I}_{32}\cup\mathcal{I}_{21}^-}\mathbf{v}_j\right)+\left(\sum_{j\in\mathcal{I}_3}\mathbf{v}_j-\sum_{j\in\mathcal{I}_{12}\cup\mathcal{I}_{22}^+}\mathbf{v}_j\right) &(4)\\&\geq q\sum_{j\in\mathcal{I}_{32}\cup\mathcal{I}_{21}^-}\mathbf{v}_j-\sum_{j\in\mathcal{I}_{12}\cup\mathcal{I}_{22}^+}\mathbf{v}_j &(5)\\&\geq-\left(q\sum_{j=1}^{|\mathcal{I}_{32}\cup\mathcal{I}_{21}^-|}t'_j-\sum_{j=|\mathcal{I}_{12}\cup\mathcal{I}_{22}^+|+1}^{n}t'_j\right)=-\langle\mathbf{v},\tilde{\mathbf{u}}\rangle,\end{aligned}$$

where Inequalities (4) and (5) hold because $\mathcal{I}_1\subseteq[k_1^*]$ and $\mathcal{I}_3\subseteq[k_2^*,\cdots,n]$.

Hence, we have $0\geq\cos\theta(\mathbf{v},\mathbf{u})\geq\cos\theta(\mathbf{v},\tilde{\mathbf{u}})$, which is equivalent to $\sin\theta(\mathbf{v},\mathbf{u})\geq\sin\theta(\mathbf{v},\tilde{\mathbf{u}})$ due to the non-negativity of $\sin\theta(\cdot,\cdot)$. □

## B Proof of Theorem 1

**Proof:** Let $\mathbf{u}$ be the random variable defined in Section 6.2, such that $\mathbf{u}_i \sim q\,\text{Bernoulli}(|\mathbf{v}_i|/q)$ for positive coordinates $\mathbf{v}_i > 0$, $\mathbf{u}_i \sim (-1)\text{Bernoulli}(|\mathbf{v}_i|)$ for negative coordinates $\mathbf{v}_i < 0$, and $\mathbf{u}_i = 0$ if $\mathbf{v}_i = 0$. For convenience, we define

$$g(x) = \begin{cases} q, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0. \end{cases}$$

We are interested in analyzing the expectation of $\mathbf{u}^T A\mathbf{u}/\mathbf{u}^T\mathbf{u}$, which is given by

$$\begin{aligned} \mathbb{E}\left[\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}}\right] &= \sum_{(k_1,k_2):1\le k_1+k_2\le n} \mathbb{E}\left[\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}} \mid \mathbf{u}^T\mathbf{u} = qk_1 + k_2\right] \mathbb{P}(\mathbf{u}^T\mathbf{u} = qk_1 + k_2) \\ &= \sum_{(k_1,k_2):1\le k_1+k_2\le n} \frac{\mathbb{E}\left[\mathbf{u}^T A\mathbf{u} \mid \mathbf{u}^T\mathbf{u} = qk_1 + k_2\right] \mathbb{P}(\mathbf{u}^T\mathbf{u} = qk_1 + k_2)}{qk_1 + k_2}. \end{aligned} \tag{6}$$

The term $\mathbb{E}\left[\mathbf{u}^T A\mathbf{u} \mid \mathbf{u}^T\mathbf{u} = qk_1 + k_2\right] \mathbb{P}(\mathbf{u}^T\mathbf{u} = qk_1 + k_2)$ in Equation (6), can be written as

$$\sum_{i\ne j} A_{i,j} g(\mathbf{v}_i) g(\mathbf{v}_j) \mathbb{P}(\mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j) \mid \mathbf{u}^T\mathbf{u} = qk_1 + k_2) \mathbb{P}(\mathbf{u}^T\mathbf{u} = qk_1 + k_2), \tag{7}$$

and using Bayes' theorem we can re-write Equation (7) as

$$\sum_{i\ne j} A_{i,j} g(\mathbf{v}_i) g(\mathbf{v}_j) \mathbb{P}(\mathbf{u}^T\mathbf{u} = qk_1 + k_2 \mid \mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j)) \mathbb{P}(\mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j)). \tag{8}$$

By Equations (6) and (8) and since $g(\mathbf{v}_i)g(\mathbf{v}_j)\mathbb{P}(\mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j)) = \mathbf{v}_i\mathbf{v}_j$, we have

$$\begin{aligned} &\sum_{(k_1,k_2):1\le k_1+k_2\le n} \frac{\sum_{i\ne j} A_{i,j}\mathbf{v}_i\mathbf{v}_j \mathbb{P}(\mathbf{u}^T\mathbf{u} = qk_1 + k_2 \mid \mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j))}{qk_1 + k_2} \\ &= \sum_{i\ne j} A_{i,j}\mathbf{v}_i\mathbf{v}_j \sum_{(k_1,k_2):1\le k_1+k_2\le n} \frac{\mathbb{P}(\mathbf{u}^T\mathbf{u} = qk_1 + k_2 \mid \mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j))}{qk_1 + k_2} \\ &= \sum_{i\ne j} A_{i,j}\mathbf{v}_i\mathbf{v}_j \mathbb{E}\left[\frac{1}{\mathbf{u}^T\mathbf{u}} \mid \mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j)\right]. \end{aligned} \tag{9}$$

As the reciprocal function is convex, we apply Jensen's inequality to Equation (9) to obtain

$$\mathbb{E}\left[\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}}\right] \ge \frac{\sum_{i\ne j} A_{i,j}\mathbf{v}_i\mathbf{v}_j}{\mathbb{E}\left[\mathbf{u}^T\mathbf{u} \mid \mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j))\right]}. \tag{10}$$

To estimate the denominator in Equation (10), we compute

$$\begin{aligned} \mathbb{E}\left[\mathbf{u}^T\mathbf{u} \mid \mathbf{u}_i = g(\mathbf{v}_i), \mathbf{u}_j = g(\mathbf{v}_j))\right] &= g(\mathbf{v}_i)^2 + g(\mathbf{v}_j)^2 + \sum_{k\ne i, k\ne j} g(\mathbf{v}_k)^2 \cdot \frac{|\mathbf{v}_k|}{|g(\mathbf{v}_k)|} \\ &\le \max\left(q\sqrt{n-2}, 2q^2 + q\frac{n-2}{\sqrt{n}}\right). \end{aligned} \tag{11}$$

Combining (10) and (12) we get

$$\mathbb{E}\left[\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}}\right] \ge \frac{\sum_{i\ne j} A_{i,j}\mathbf{v}_i\mathbf{v}_j}{\max\left(q\sqrt{n-2}, 2q^2 + q\frac{n-2}{\sqrt{n}}\right)} = \frac{\lambda_1(A)}{\max\left(q\sqrt{n-2}, 2q^2 + q\frac{n-2}{\sqrt{n}}\right)}. \tag{12}$$

Hence, the expected approximation ratio is

$$\mathcal{O}(q\sqrt{n})\mathbb{E}\left[\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}}\right] \ge \lambda_1(A) \ge OPT,$$

where $OPT$ is the optimum of MAX-DRQ. □

## C Proof of Lemma 2

**Proof:** Consider a graph $G = (V, E)$ consisting of $|V| = n = 2c+1$ nodes, for some $c \geq 1$, where $2c$ nodes form a negative clique and the extra node $v$ is negatively connected to $c$ of the nodes in the clique. Let $A$ be the signed adjacency matrix of $G$. We will show the problem instance defined on $G$ results in an optimal value of MAX-DRQ equal to $OPT = \mathcal{O}(1)$, while $\lambda_1(A)$ is $\Omega(\sqrt{n})$.

Any solution $\mathbf{u} \in \{0, -1, q\}^n$ to MAX-DRQ defines the two sets $S_p = \{i : \mathbf{u}_i = q\}$ and $S_n = \{i : \mathbf{u}_i = -1\}$. We claim that $\max_{\mathbf{u}\in\{0,-1,q\}^n} \mathbf{u}^T A\mathbf{u}/\mathbf{u}^T\mathbf{u} \leq 2$, and will show it by considering 3 cases:

(Case 1) $v \notin S_p \cup S_n$:

$$\begin{aligned}\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}} &= \frac{-q^2\overbrace{|S_p|(|S_p|-1)}^{2|E(S_p)|} + q\overbrace{2|S_p||S_n|}^{2|E(S_p,S_n)|} - \overbrace{|S_n|(|S_n|-1)}^{2|E(S_n)|}}{q^2|S_p| + |S_n|} \\ &= \frac{-(q|S_p| - |S_n|)^2 + q^2|S_p| + |S_n|}{q^2|S_p| + |S_n|}. \end{aligned} \tag{13}$$

Let $r = q|S_p| - |S_n|$ and let $\epsilon = r/q|S_p| \leq 1$. Then, Equation (13) can be written as

$$\begin{aligned}\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}} &= \frac{q(q+1)|S_p| - r(r+1)}{|S_p|q(q+1) - r} \\ &= \frac{(q+1) + \frac{1}{4(q|S_p|)}}{(q+1)-\epsilon} - \frac{(r+\frac{1}{2})^2}{q|S_p|(q+1-\epsilon)} \\ &\leq \frac{(q+1) + \frac{1}{4(q|S_p|)}}{(q+1)-\epsilon} \\ &\leq \frac{q+2}{q} \leq 2 = \mathcal{O}(1).\end{aligned}$$

(Case 2) $v \in S_p$:

$$\begin{aligned}\mathbf{u}^T A\mathbf{u} = &-q^2\left(\underbrace{(|S_p|-1)(|S_p|-2)}_{2|E(S_p\setminus\{v\})|} + 2|E(\{v\}, S_p)|\right) \\ &+ q\left(\underbrace{2(|S_p|-1)|S_n|}_{2|E(S_p\setminus\{v\},S_n)|} + 2|E(\{v\}, S_n)|\right) - \underbrace{|S_n|(|S_n|-1)}_{2|E(S_n)|} \\ = &-(q(|S_p|-1) - |S_n|)^2 + |S_n| \\ &+ q^2(|S_p|-1) + 2q|E(\{v\}, S_n)| - 2q^2|E(\{v\}, S_p)| \\ \leq &-(q(|S_p|-1) - |S_n|)^2 + |S_n| + q^2(|S_p|-1) + 2q|S_n|.\end{aligned} \tag{14}$$

Let $r = q(|S_p|-1) - |S_n|$ and write Equation (14) as

$$\begin{aligned}\mathbf{u}^T A\mathbf{u} &= -(r-q)^2 + q(q+3)(|S_p|-1) + q^2 - r \\ &\leq q(q+3)(|S_p|-1) + q^2 - r.\end{aligned} \tag{15}$$

By (15) and letting $\epsilon = r/q(|S_p|-1) \leq 1$, we have

$$\begin{aligned}\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}} &\leq \frac{q(q+3)(|S_p|-1) + q^2 - r}{q(q+1)(|S_p|-1) + (q^2 - r)} \\ &= 1 + \frac{2}{(q+1) + (q/(|S_p|-1) - \epsilon)} \leq 2 = \mathcal{O}(1).\end{aligned}$$

(Case 3) $v \in S_n$:

$$\begin{aligned}\mathbf{u}^T A\mathbf{u} &= q^2|S_p|(|S_p|-1) + q\left(\underbrace{|S_p|(|S_n|-1)}_{2|E(S_n\setminus\{v\},S_p)|} + 2|E(\{v\},S_p)|\right) \\ &\quad - \left(\underbrace{(|S_n|-1)(|S_n|-2)}_{2|E(S_n\setminus\{v\})|} + 2|E(\{v\},S_n)|\right) \\ &= -\left(q|S_p| - (|S_n|-1)\right)^2 + q^2|S_p| + |S_n| - 1 \\ &\quad + 2q|E(\{v\},S_p)| - 2|E(\{v\},S_n)| \\ &\leq -\left(q|S_p| - (|S_n|-1)\right)^2 + q^2|S_p| + |S_n| - 1 + 2q|S_p|. \end{aligned} \tag{16}$$

Let $r = q|S_p| - (|S_n|-1)$ and write Inequality (16) as

$$\begin{aligned}\mathbf{u}^T A\mathbf{u} &= -(r+\frac{1}{2})^2 + q(q+3)|S_p| + \frac{1}{4} \\ &\leq q(q+3)|S_p| + \frac{1}{4}. \end{aligned} \tag{17}$$

By (17) and letting $\epsilon = (r+1)/q|S_p| \leq 1$, we have

$$\begin{aligned}\frac{\mathbf{u}^T A\mathbf{u}}{\mathbf{u}^T\mathbf{u}} &\leq \frac{q(q+3)|S_p| + \frac{1}{4}}{q(q+1)|S_p| - (r+1)} \\ &= 1 + \frac{2 + 1/(4q|S_p|) + \epsilon}{(q+1) - \epsilon} \\ &\leq 2 = \mathcal{O}(1).\end{aligned}$$

Therefore, we know that the optimal solution $OPT$ of MAX-DRQ is $\mathcal{O}(1)$. However, consider a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{x} = \left[\sqrt{\frac{n+1}{2n}}, \underbrace{\frac{1}{\sqrt{2n}}, \cdots, \frac{1}{\sqrt{2n}}}_{c \text{ entries}}, \underbrace{\frac{-1}{\sqrt{2n}}, \cdots, \frac{-1}{\sqrt{2n}}}_{c \text{ entries}}\right], \tag{18}$$

where the first entry of $\mathbf{x}$ corresponds to $v$. Then, the vector $\mathbf{x}$ defined in Equation (18) gives

$$\begin{aligned}\frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} &= \frac{\sqrt{n+1}(n-1)}{2n} + \frac{n-1}{2n} \\ &= \frac{\sqrt{n+1}+1}{2} - \frac{\sqrt{n+1}+1}{2n} \\ &= \Omega(\sqrt{n}).\end{aligned}$$

As $\lambda_1(A) = \max_{\mathbf{x}\in\mathbb{R}^n\setminus\{0\}} \mathbf{x}^T A\mathbf{x}/\mathbf{x}^T\mathbf{x}$, we have shown $\lambda_1(A) \geq OPT \cdot \Omega(\sqrt{n})$. $\square$

## D Experiment Results

### D.1 Dataset

WoW-EP8 [1] is the interaction network of authors in the 8th legislature of the EU Parliament, where edge signs indicate if two authors are collaborative or competitive to each other. Bitcoin [4] is the trust-distrust network of users trading on the Bitcoin OTC platform. WikiVot [4] collects the positive and negative votes for electing Wikipedia admins. Referendum [3] collects the tweets about the Italian constitutional referendum in 2016, and edge signs indicate if two users are classified to have the same stance or not. Slashdot [4] is a friend-foe network collected from the Slashdot Zoo feature. WikiCon [2] collects the positive and negative iterations of users editing the English Wikipedia. Epinions [4] is the trust-distrust network of users on the online social network Epinions. WikiPol [5] is the interaction network of users who have edited the English Wikipedia pages about politics.

### D.2 Execution Time

Table 2: Running times for the results shown in Table 1. All times are shown in seconds. Dashes indicate that a method cannot finish execution due to memory limit exceeded.

| | | WoW-EP8 | Bitcoin | WikiVot | Referendum | Slashdot | WikiCon | Epinions | WikiPol |
|---|---|---|---|---|---|---|---|---|---|
| | $\|V\|$ | 790 | 5 881 | 7 115 | 10 884 | 82 140 | 116 717 | 131 580 | 138 587 |
| | $\|E\|$ | 116 009 | 21 492 | 100 693 | 251 406 | 500 481 | 2 026 646 | 711 210 | 715 883 |
| | $\|E_-\|/\|E\|$ | 0.2 | 0.2 | 0.2 | 0.1 | 0.2 | 0.6 | 0.2 | 0.1 |
| $k=2$ | SCG-MA | 2 | 1 | 2 | 4 | 10 | 217 | 109 | 25 |
| | SCG-MO | 2 | 1 | 2 | 4 | 11 | 70 | 94 | 15 |
| | SCG-B | 13 | 9 | 21 | 44 | 693 | 3 584 | 1 906 | 1 624 |
| | SCG-R | 4 | 3 | 6 | 17 | 70 | 485 | 37 | 217 |
| | KOCG | 3 | 11 | 16 | 25 | 1 243 | 3 269 | 3 208 | 3 506 |
| | BNC-$k$ | 2 | 1 | 2 | 4 | — | — | — | — |
| | BNC-$(k+1)$ | 2 | 1 | 2 | 4 | — | — | — | — |
| | SPONGE-$k$ | 2 | 5 | 3 | 4 | — | — | — | — |
| | SPONGE-$(k+1)$ | 2 | 11 | 4 | 9 | — | — | — | — |
| $k=6$ | SCG-MA | 3 | 1 | 6 | 16 | 75 | 394 | 132 | 136 |
| | SCG-MO | 3 | 1 | 6 | 18 | 74 | 229 | 107 | 139 |
| | SCG-B | 17 | 29 | 78 | 201 | 3 280 | 10 637 | 5 455 | 5 714 |
| | SCG-R | 3 | 5 | 9 | 21 | 118 | 415 | 219 | 892 |
| | KOCG | 1 | 5 | 8 | 14 | 690 | 1 837 | 1 845 | 1 724 |
| | BNC-$k$ | 2 | 1 | 2 | 4 | — | — | — | — |
| | BNC-$(k+1)$ | 2 | 1 | 2 | 4 | — | — | — | — |
| | SPONGE-$k$ | 2 | 7 | 6 | 20 | — | — | — | — |
| | SPONGE-$(k+1)$ | 2 | 5 | 4 | 26 | — | — | — | — |

### D.3 Detected Group Sizes

Figure 2, extracted from the Referendum dataset, shows the typical distribution of the group sizes for all the comparison methods. This pattern is similar to all other datasets except WoW-EP8. That is, SCG-MA, SCG-MO, and SCG-R return the largest groups while KOCG-top-1, BNC-$(k+1)$, and SPONGE-$(k+1)$ return the smallest groups.

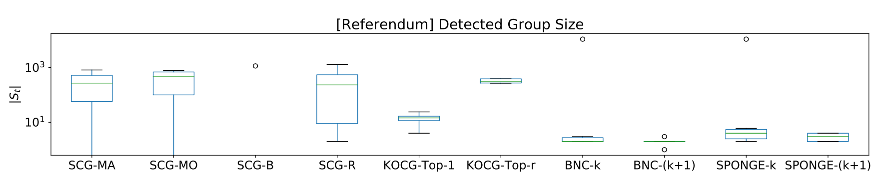



Figure 2: The typical group size distribution on all the datasets except WoW-EP8 when $k=6$.

On the other hand, WoW-EP8 shows a different group-size distribution, which is shown in Figure 3. All SCG methods and BNC-$k$ find one giant group. By checking the polarity (Table 1 in main paper), their scores are high, so this probably suggests there exists a giant conflicting group in the network.

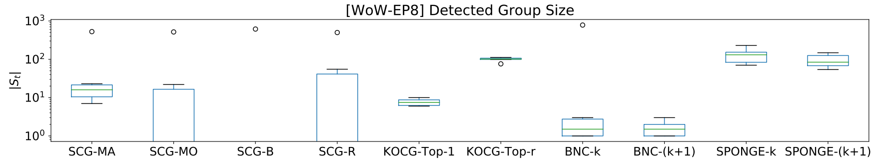



Figure 3: The group size distribution on `WoW-EP8` when $k = 6$.

### D.4 Deciding $k$

We present a heuristic similar to Elbow Method [6] to decide $k$, which consists of the following steps:

1. Run SCG multiple times with different $k$.
2. Draw a `DRQ-Plot`, where the Discrete Rayleigh Quotient (DRQ) values in each run are sorted, and then plot the $i$-th largest DRQ value at the $i$-th location.
3. Decide $k$ to be one of the "knees" of the curve.

The reason why the heuristic works is that, if there exist conflicting groups and the noise-level is not too high, then the leading eigenvector should be indicative of the true conflicting groups and have large DRQ values in the first $k-1$ iterations, while the leading eigenvector only captures noise structures and has low DRQ value after the $k$-th iteration. Therefore, it is expected to see knees of the curve at the $(k-1)$-th iteration.

First, we evaluate the heuristic using m-SSBM under the same setting ($k = 6$, $\ell = 100$, and $n = 2\,000$) by varying $\eta = 0 : 0.1 : 0.6$. The result of detecting the conflicting groups by SCG-MA is depicted in Figure 4. As expected, the most prominent knee is at the 5-th iteration when the noise-level is not too high ($\eta \leq 0.3$). As the noise-level increases ($\eta \geq 0.4$), the knee at the 5-th iteration becomes less obvious and some artificial knees that fit the random noise emerge.

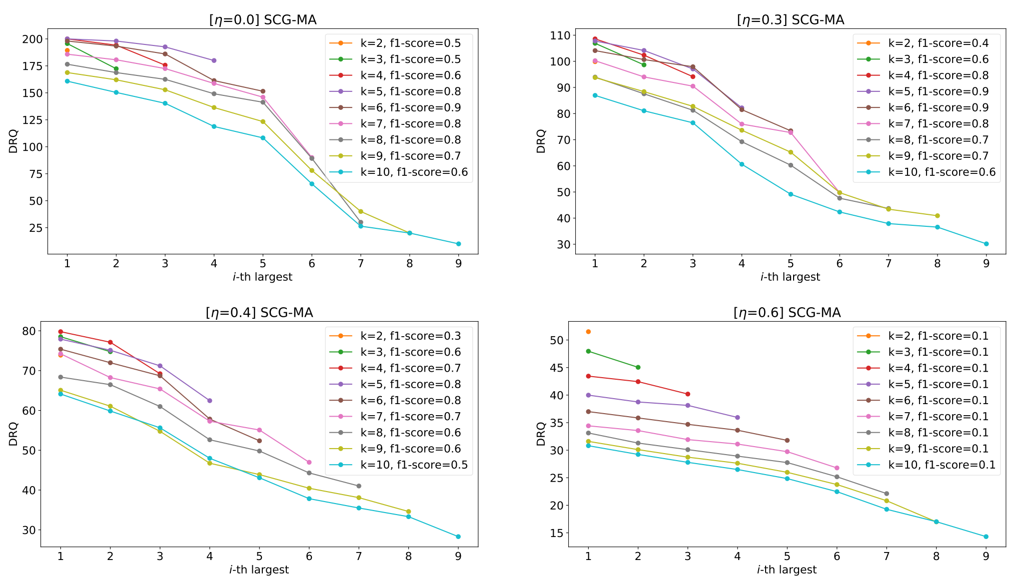
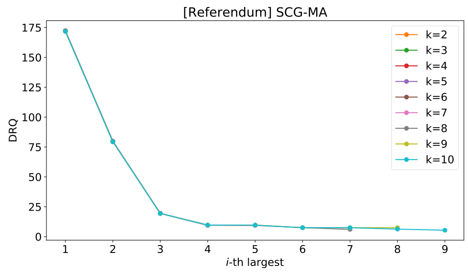



Figure 4: Run SCG-MA with different $k$ on networks generated by m-SSBM ($k = 6$, $\ell = 100$, and $n = 2\,000$). Each setting is repeated 20 times and reported the average.

Finally, we use the heuristic on the real-world datasets to decide $k$ and show the result in Figure 5. Our analysis suggests that `Referendum` has 4 conflicting groups, because the most prominent knee appears at the 3-th iteration, while on `Epinions`, there are two prominent knees at the 3rd and the 4-th iterations, so there are probably 4 or 5 conflicting groups in the network.



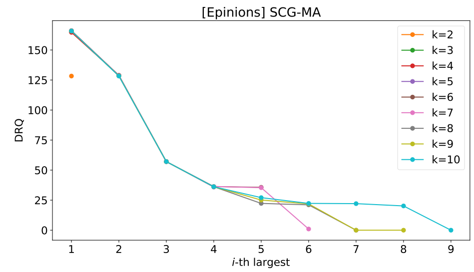




Figure 5: Run SCG-MA Real-world networks with different $k$.

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