(Probably) Concave Graph Matching Supplementary material

Haggai Maron Weizmann Institute of Science Rehovot, Israel haggai.maron@weizmann.ac.il

Yaron Lipman Weizmann Institute of Science Rehovot, Israel yaron.lipman@weizmann.ac.il

1 Frank-Wolfe with concave search

An orthogonal basis to $\text{lin}(\mathcal{F})$ is computed similarly to Lemma 1 in the paper:

Lemma 1 (orthonormal basis for one-sided permutations). *If the columns of* $F \in \mathbb{R}^{n_0 \times (n_0 - 1)}$ *form* an orthonormal basis for $\mathbf{1}^\perp$ in \mathbb{R}^{n_0} then the columns of $F \otimes I_n$ are an orthonormal basis for $\text{lin}(\mathcal{F})$.

The energy $E_2(X)$ in this case does not model the matching problem well since it gives rise to trivial solutions. Instead, we chose to optimize a similar energy ([Solomon et al.,](#page-0-0) [2016](#page-0-0)): $E(X) =$ $\sum_{ijkl} X_{ij} X_{kl} (A_{ik} − B_{jl})^2$. This energy can also be written in matrix form: $[X]^T M [X]$ where $M = -2B \otimes A + 11^T \otimes A^2 + B^2 \otimes 11^T$ (where C^2 implies entry-wise operation) and after restricting it to lin(*F*) its Hessian is of the form *−*2*F BF ⊗ A* + *F B.*2*F ⊗* 11*^T* . Assuming *A, B* are Euclidean distance matrices, the right summand is negative semidefinite, but the left summand is not. This is because that *A* is not conjugated by *F*: it has a large positive eigenvalue as a result of the Perron-Frobenius Theorem.

The linear program solved in each iteration of the algorithm takes a surprisingly simple form: it amounts to solving $\min_{X \in \text{hull}(\mathcal{F})}$ tr($\nabla E(X_0)^T X$) which can be solved simply by assigning the value 1 to the index of the minimal value in each row of $\nabla E(X_0)$. This procedure always outputs solutions in *F*.

The convex energies we subtract from the objective during the concave search should be constant on $\mathcal F$ so a reduction in the subtracted energy is the same as in the original energy $E(X)$. We use the quadratic form defined by $\lambda * \Lambda$ where Λ is a $nn_0 \times nn_0$ diagonal matrix defined by $D_{ijij} =$ $\max_j {\sum_{kl} |M_{ijkl}|}.$ *D* is a positive definite matrix and for $\lambda = 1$, $W - D$ is guaranteed to be negative semidefinite. The values of *λ* need not be discretized since there are only *n* different critical values - the ones that change the minimum calculation mentioned in the previous paragraph.

References

Solomon, J., Peyré, G., Kim, V. G., and Sra, S. (2016). Entropic metric alignment for correspondence problems. *ACM Transactions on Graphics (TOG)*, 35(4):72.