
Supplement for: Algebraic tests of general Gaussian latent tree models

Dennis Leung

Department of Data Sciences and Operations
University of Southern California
dmhleung@uw.edu

Mathias Drton

Department of Statistics, University of Washington &
Department of Mathematical Sciences, University of Copenhagen
md5@uw.edu

In this supplement we furnish proofs for the main text of “Algebraic tests of general Gaussian latent tree models”.

6 Proof of Corollary 2.2

We only sketch the proof here since it is exactly analogous to that of Theorem 3 in Shiers et al. [2016]. First, consider the special case where all the entries of Σ , and hence the the Pearson correlations ρ_{pq} , $1 \leq p \neq q \leq m$, are strictly positive. In this case condition (i)(a) is redundant. Via the isomorphism

$$\delta_{pq} = -\log \rho_{pq},$$

between the parametrizations in (2.1) and all T -induced pseudometrics, the discussion preceding our corollary readily translates (2.2) into (i)(c), (ii)(b), (ii)(c) and (2.3) into (ii)(a), whereas the triangular inequality property of pseudometrics is translated into (i)(b) for triples $\{p, q, r\}$ that are not in \mathcal{L} . The general case of Σ with nonzero but not necessarily positive entries is then addressed by incorporating condition (i)(a).

7 Proof of Lemma 2.1

To prove the lemma, we first collect all the required graphical notions borrowed from Semple and Steel [2003]. We attempted to make this proof as self-contained as possible, but the readers are encouraged to read Semple and Steel [2003] for more background on mathematical phylogenetics.

Suppose we are given a tree $T = (V, E)$. If \tilde{V} is a subset of V , $T(\tilde{V})$ denotes the minimal subtree of T that contains all the nodes in \tilde{V} . If $e \in E$, $T \setminus e$ is the graph obtained by removing e , and T/e is the tree obtained from T by identifying the ends of e and then deleting e . In particular, if $v \in V$ is a node of degree two and e is an edge incident with v , T/e is said to be obtained from T by *suppressing* v . If $v_1, v_2 \in V$, $ph_T(v_1, v_2)$ is the set of edges on the unique path connecting v_1 and v_2 .

We will also need the notion of an \mathbf{X} -tree. An \mathbf{X} -tree, or semi-labeled tree on a set \mathbf{X} , is an ordered pair $\mathcal{T} = (T, \phi)$, where T is a tree with node set V and $\phi : \mathbf{X} \rightarrow V$ is a (labeling) map with the property that, for each $v \in V$ of degree at most two, $v \in \phi(\mathbf{X})$. Note that ϕ is not necessarily injective. Moreover, if \mathbf{X}' is a subset of \mathbf{X} , $T|\mathbf{X}'$ is the tree obtained from $T(\phi(\mathbf{X}'))$ by suppressing all the nodes of degree two that are not in $\phi(\mathbf{X}')$. We then define the *restriction of \mathcal{T} to \mathbf{X}'* , denoted $\mathcal{T}|\mathbf{X}'$, to be the \mathbf{X}' -tree $(T|\mathbf{X}', \phi|\mathbf{X}')$.

Finally, we introduce the notion of \mathbf{X} -split. For a set \mathbf{X} , an \mathbf{X} -split is a partition of \mathbf{X} into two non-empty sets. We denote the \mathbf{X} -split whose blocks are A and B by $A|B$ where the order of A and

B in the notation doesn't matter. Now suppose $\mathcal{T} = (T, \phi)$ is an \mathbf{X} -tree with an edge set E . For each $e \in E$, $T \setminus e$ must consist of two components V_1^e and V_2^e which induce an \mathbf{X} -split $\phi^{-1}(V_1^e) | \phi^{-1}(V_2^e)$. We then define $\Sigma(\mathcal{T}) := \{\phi^{-1}(V_1^e) | \phi^{-1}(V_2^e) : e \in E\}$ as the collection of all \mathbf{X} -splits induced by \mathcal{T} .

Important remark: In all the definitions above, \mathbf{X} is not specified as a subset of the node set V for a given tree. Nonetheless, when we have a tree $T = (V, E)$ with a subset of observed nodes $\mathbf{X} \subset V$ as in the main text, we will slightly abuse the notations by identifying T with the \mathbf{X} -tree whose labeling map is simply the identity function. Moreover, if $\mathbf{X}' \subset \mathbf{X}$, we will also identify $T|_{\mathbf{X}'}$ with the \mathbf{X}' -tree that is the restriction of T (as an \mathbf{X} -tree) to \mathbf{X}' .

Now we begin to prove Lemma 2.1. The “only if” part of the theorem is trivial and we will only prove the “if” part of the statement.

We recall that $\mathbf{X} = \{X_1, \dots, X_m\}$. Let δ be a pseudo-metric on \mathbf{X} satisfying the two conditions (2.2) and (2.3) in display. For any four distinct points $p, q, r, s \in [m]$, given the tree structure of T it must be true that $ph_T(\pi_p, \pi_q) \cap ph_T(\pi_r, \pi_s) = \emptyset$ for some permutation π of p, q, r, s . By (2.2), together with the fact that δ is a pseudo-metric, δ is in fact a *tree metric* (Semple and Steel [2003, Theorem 7.2.6]), i.e., there exists an \mathbf{X} -tree $\tilde{T} = (\tilde{T}, \tilde{\phi})$ for a tree $\tilde{T} = (\tilde{V}, \tilde{E})$ and a labeling map $\tilde{\phi} : \mathbf{X} \rightarrow \tilde{V}$, as well as a *strictly positive* weighting function $\tilde{w} : \tilde{E} \rightarrow \mathbb{R}_{>0}$ such that

$$\delta_{pq} = \begin{cases} \sum_{\tilde{e} \in ph_{\tilde{T}}(\tilde{\phi}(X_p), \tilde{\phi}(X_q))} \tilde{w}(\tilde{e}) & \text{if } \tilde{\phi}(X_p) \neq \tilde{\phi}(X_q) \\ 0 & : \tilde{\phi}(X_p) = \tilde{\phi}(X_q) \end{cases}$$

for all $p, q \in [m]$. By Theorem 6.3.5(i) and Lemma 7.1.4 in Semple and Steel [2003], to show that δ can be induced from T it suffices to show that for any $\mathbf{X}' \subset \mathbf{X}$ of size at most 4, the two restricted \mathbf{X}' -trees $\tilde{T}|_{\mathbf{X}'}$ and $T|_{\mathbf{X}'}$ are such $\Sigma(\tilde{T}|_{\mathbf{X}'}) \subset \Sigma(T|_{\mathbf{X}'})$. Note that this is trivial for $|\mathbf{X}'| = 1$ and $|\mathbf{X}'| = 2$. For $3 \leq |\mathbf{X}'| \leq 4$, we first note that

$$\{X_p\} | \mathbf{X}' \setminus \{X_p\} \in \Sigma(\tilde{T}|_{\mathbf{X}'}) \text{ if and only if } \delta_{pq} + \delta_{pr} - \delta_{qr} > 0 \text{ for all } X_q, X_r \in \mathbf{X}' \setminus \{X_p\} \quad (7.1)$$

and

$$\{X_p, X_q\} | \{X_r, X_s\} \in \Sigma(\tilde{T}|_{\mathbf{X}'}) \text{ if and only if } \delta_{pr} + \delta_{qs} - \delta_{pq} - \delta_{rs} > 0 \text{ (and } \delta_{ps} + \delta_{qr} - \delta_{pq} - \delta_{rs} > 0). \quad (7.2)$$

These characterization for the elements in $\Sigma(\tilde{T}|_{\mathbf{X}'})$ can be easily checked; also see Semple and Steel [2003, p.148] where these characterizations are stated. To finish the proof it remains to show that, when $3 \leq |\mathbf{X}'| \leq 4$, any \mathbf{X}' -split $\{X_p\} | \mathbf{X}' \setminus \{X_p\}$ as in (7.1) or any \mathbf{X}' -split $\{X_p, X_q\} | \{X_r, X_s\}$ as in (7.2) must also be an element of $\Sigma(T|_{\mathbf{X}'})$.

First, towards a contradiction, suppose there exists an \mathbf{X}' -split $\{X_p\} | \mathbf{X}' \setminus \{X_p\}$ that is an element of $\Sigma(\tilde{T}|_{\mathbf{X}'})$ but not an element of $\Sigma(T|_{\mathbf{X}'})$. Since $\{X_p\} | \mathbf{X}' \setminus \{X_p\}$ is not an element of $\Sigma(T|_{\mathbf{X}'})$, by considering $T|_{\mathbf{X}'}$ as a tree it must be the case that the node X_p has degree at least two. Then, by condition (2.3), there must exist two distinct X_q and X_r in the set $\mathbf{X}' \setminus \{X_p\}$ such that $\delta_{pq} + \delta_{pr} = \delta_{qr}$. But this reaches a contradiction since by (7.1) $\delta_{pq} + \delta_{pr} - \delta_{qr} > 0$ as $\{X_p\} | \mathbf{X}' \setminus \{X_p\} \in \Sigma(\tilde{T}|_{\mathbf{X}'})$.

Similarly, suppose $\{X_p, X_q\} | \{X_r, X_s\}$ is an element of $\Sigma(\tilde{T}|_{\mathbf{X}'})$ but not an element of $\Sigma(T|_{\mathbf{X}'})$. Since $\{X_p, X_q\} | \{X_r, X_s\} \in \Sigma(\tilde{T}|_{\mathbf{X}'})$, the two strict inequalities in (7.2) must be true. On the other hand, if $T|_{\mathbf{X}'}$ has any of the configurations in Figure 2.2(a) – (c), since $\{X_p, X_q\} | \{X_r, X_s\} \notin \Sigma(T|_{\mathbf{X}'})$ it must be true that $ph_T(p, q) \cap ph_T(r, s) \neq \emptyset$, in which case it must lead to either $ph_T(p, r) \cap ph_T(q, s) = \emptyset$ or $ph_T(p, s) \cap ph_T(q, r) = \emptyset$, contradicting one of the inequalities in (7.2) by condition (2.2). If $T|_{\mathbf{X}'}$ has the configuration in Figure 2.2(d) or (e), then it must be the case that both $ph_T(p, r) \cap ph_T(q, s)$ and $ph_T(p, s) \cap ph_T(q, r)$ are empty sets, which also contradict both inequalities in (7.2) by condition (2.2).

References

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