Supplementary Material for Causal Discovery from Discrete Data using Hidden Compact Representation

Ruichu Cai [1], Jie Qiao[1], Kun Zhang[2], Zhenjie Zhang[3], Zhifeng Hao[1, 4]
[1] School of Computer Science, Guangdong University of Technology, China
[2] Department of philosophy, Carnegie Mellon University
[3] Singapore R&D, Yitu Technology Ltd.
[4] School of Mathematics and Big Data, Foshan University, China
cairuichu@gdut.edu.cn, qiaojie.chn@gmail.com, kunz1@andrew.cmu.edu,
zhenjie.zhang@yitu-inc.com, zfhao@gdut.edu.cn

Theorem 2. *Assume that in the causal direction there exists the transformation $Y' = f(X)$ such that $P(Y|X) = P(Y|Y')$, where $|Y'| < |X|$, and and assumption A1 holds. Then to produce the same distribution $P(X, Y)$, the reverse direction must involve more effective number of parameters in the model than the causal direction.*

Proof. Recall that the parameter for the causal direction is $d_{X\to Y} = |X| - 1 + |Y'|(|Y| - 1)$. Since $|Y'| < |X|$, we conclude $d_{X\to Y} < |X| - 1 + |X|(|Y| - 1)$. For reverse direction, the number of effective parameters is $d_{Y\to X} = |Y| - 1 + |X'|(|X| - 1)$. Based on Theorem 1, the reverse direction does not admit a low-cardinality hidden representation. Hence, we have $|X'| = |Y|$. Compare the difference between $d_{X\to Y}$ and $d_{Y\to X}$, we have:

$$\begin{aligned} d_{X\to Y} - d_{Y\to X} &< |X| - 1 + |X|(|Y| - 1) - (|Y| - 1 + |Y|(|X| - 1)) \\ &= 0 \end{aligned} \tag{1}$$

Thus, reverse direction must involve more effective number of parameters than the causal direction. □

Theorem 3. *If the reverse direction involves more parameters than the causal direction to produce the same distribution $P(X, Y)$, the BIC of the causal direction is asymptotically higher than that of the reverse one.*

Proof. First, we will proof that the likelihood on the causal direction asymptotically greater or equal than that of the reverse direction. Equation (2) gives the gap between the causal direction M and the reverse direction $\hat{M}$.

$$\begin{aligned} &\mathcal{L}(\mathcal{D}; M) - \mathcal{L}(\mathcal{D}; \hat{M}) \\ =&\sum_{i=1}^{m} \log \frac{P(X = x_i)P(Y = y_i|Y' = f(x_i))}{P(Y = y_i)P\left(X = x_i|X' = \hat{f}(y_i)\right)} \\ =&\sum_{i=1}^{m} \log \frac{P(X = x_i, Y = y_i)}{P(Y = y_i)P\left(X = x_i|X' = \hat{f}(y_i)\right)} \\ =&mE_{x,y\sim p(x,y)}\left(\log \frac{P(X = x, Y = y)}{P(Y = y)P\left(X = x|X' = \hat{f}(y)\right)}\right) \\ =&mKL\left(P(X = x, Y = y)||P(Y = y)P\left(X = x|X' = \hat{f}(y)\right)\right) \\ \geqslant&0 \end{aligned} \tag{2}$$

32nd Conference on Neural Information Processing Systems (NeurIPS 2018), Montréal, Canada.

The first equality is based on the likelihood of the HCR model. The second equality is based on the fact that $X \perp\!\!\!\perp Y | Y'$. The third equality is based on the existing of sufficient number of samples. The forth equality and the fifth inequality are based on the definition and the property of KL-divergence respectively.

Second, we will show that the penalty on the causal direction less than the reverse direction. Because the causal direction M and the reverse direction $\hat{M}$ produce the same probability $P(X, Y)$, and the inverse direction $\hat{M}$ involves more parameters than the causal direction M according to the Theorem 2.

Thus, the causal direction's BIC is asymptotically higher than the reverse ones. $\square$