A Negative association properties of ESR measures

Proposition 1. There exist ESR measures that are not SR.

Proof. Recall [11, Thm. 4.1] that a real bivariate affine polynomial $p(x, y)$ is stable if and only if

$$\partial_x p \partial_y p - p\, \partial_{xy} p \geq 0.$$

For an E-DPP with a kernel $L \in \mathbb{R}^{2\times 2}$, this is exactly $l_{11}^p l_{22}^p \geq \det(L)^p$, which is clearly true for all $p \geq 0$ as L must be positive semi-definite.

However, stability does not hold in general. To obtain a counterexample, consider the generating polynomial $p(x, y, z)$ for an E-DPP of dimension 3 (writing $d = \det(L)$ as a shorthand):

$$p(x, y, z) := d^p + l_{11}^p yz + l_{22}^p xz + l_{33}^p xy + \det(L[1, 2])^p z + \det(L[1, 3])^p y + \det(L[2, 3])^p x + xyz.$$

If $p(x, y, z)$ is SR, p must be stable through conditioning [11, Theorem 4.1]; hence, $p(x, y, 1)$ must also be stable. Writing $d_{ij} = \det(L[i, j])$, this requires that

$$p(x, y, 1) = d^p + d_{12}^p + (d_{22}^p + d_{23}^p)x + (d_{11}^p + d_{13}^p)y + (d_{33}^p + 1)xy$$

be stable. Since $p(x, y, 1)$ is a real bivariate affine polynomial, we must then have

$$(d_{22}^p + d_{23}^p)(d_{11}^p + d_{13}^p) \geq (d^p + d_{12}^p)(d_{33}^p + 1).$$

Finally, one can verify that this last inequality is easily violated for several choices of (non-diagonal) positive semi-definite matrices. □

Theorem 1. There exists $\epsilon > 0$ such that for any $p \in [1 - \epsilon, 1 + \epsilon]$ and $n \in \mathbb{N}$, the set $\{L \in \mathbb{R}^{n\times n} : \text{E-DPP}(L, p) \text{ is SR}\}$ is strictly greater than the set of block-diagonal matrices with 2×2 blocks.

Proof. We write $d_S^i = \det(L[S \cup \{i\}])^p$ and when possible $d_{ijk} = \det(L[\{i, j, k\}])^p$.

Let $L \succeq 0 \in \mathbb{R}^{n\times n}$. The associated E-DPP is SR if and only if $P_{ij}(z) \geq 0$ where $P_{ij}(z)$ is defined for any $1 \leq i \neq j \leq n$ as

$$\begin{aligned}
P_r(z) &= \frac{\partial f}{\partial z_i}(z)\frac{\partial f}{\partial z_j}(z) - f(z)\frac{\partial^2 f}{\partial z_i \partial z_j}(z) \\
&= \Big(\sum_{S\in\mathcal{Y}'} d_S^i z^S\Big)\Big(\sum_{S\in\mathcal{Y}'} d_S^j z^S\Big) - \Big(\sum_{S\in\mathcal{Y}'} d_S^{ij} z^S\Big)\Big(\sum_{S\in\mathcal{Y}'} d_S z^S\Big).
\end{aligned}$$

where we write $\mathcal{Y}' = \{S \in [n], i \notin S, j \notin S\}$ and $z = (z_1, \ldots, z_n)$ where components z_i and z_j are removed. Now, choose $k \in [n]\backslash\{i, j\}$, and write $\tilde{z}$ the vector z without component z_k. Write also $\mathcal{Y}'_k = \{S \in \mathcal{Y}', k \notin S\}$. Then,

$$\begin{aligned}
P_{ij}(z) =& (\sum_{k\notin S} d_S^i \tilde{z}^S + z_k \sum_{k\in S} d_S^i \tilde{z}^S)(\sum_{k\notin S} d_S^j \tilde{z}^S + z_k \sum_{k\in S} d_S^j \tilde{z}^S) \\
&- (\sum_{k\notin S} d_S^{ij} \tilde{z}^S + z_k \sum_{k\in S} d_S^{ij} \tilde{z}^S)(\sum_{k\notin S} d_S \tilde{z}^S + z_k \sum_{k\in S} d_S \tilde{z}^S) \\
=& \left(\sum_{S\in\mathcal{Y}'_k} d_S^{ik} \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S^{jk} \tilde{z}^S - \sum_{S\in\mathcal{Y}'_k} d_S^{ijk} \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S^k \tilde{z}^S\right) z_k^2 \\
&+ \left(\sum_{S\in\mathcal{Y}'_k} d_S^{ik} \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S^j \tilde{z}^S + \sum_{S\in\mathcal{Y}'_k} d_S^i \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S^{jk} \tilde{z}^S \right. \\
&\qquad \left. - \sum_{S\in\mathcal{Y}'_k} d_S^{ijk} \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S \tilde{z}^S - \sum_{S\in\mathcal{Y}'_k} d_S^{ij} \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S^k \tilde{z}^S\right) z_k \\
&+ \left(\sum_{S\in\mathcal{Y}'_k} d_S^i \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S^j \tilde{z}^S - \sum_{S\in\mathcal{Y}'_k} d_S^{ij} \tilde{z}^S \sum_{S\in\mathcal{Y}'_k} d_S \tilde{z}^S\right) \\
=& A z_k^2 + B z_k + C
\end{aligned}$$

Hence, L yields a SR E-DPP measure if and only if the following inequalities hold for all i, j, k:

$$(B/2)^2 \leq AC; \quad A \geq 0; \quad C \geq 0. \tag{A.1}$$

When $n = 3$, Eq. (A.1) reduces to the following arithmetic-geometric inequality, as $\mathcal{Y}'_k = \{\emptyset\}$:

$$\left(\frac{d_i d_{jk} + d_j d_{ik} - d_{ij} d_k - d_{ijk}}{2}\right)^2 \leq (d_i d_j - d_{ij})(d_{jk} d_{ik} - d_k d_{ijk}) \tag{A.2}$$

One can easily obtain 3 positive semi-definite matrices L which verify Eq. (A.2) strictly for $p = 1$; in particular, by continuity, there exists $\epsilon > 0$ such that the E-DPP generated by the kernel L and power $p \in [1 - \epsilon, 1 + \epsilon]$ will still verify Eq. (A.2).

Then, as a block-diagonal matrix such that each diagonal block yields SR E-DPPs also yields a SR E-DPP, we can thus generate block-diagonal matrices of any size n such that the blocks are either L or 2×2 matrices, which all yield SR E-DPPs for $p \in [1 - \epsilon, 1 + \epsilon]$. □

B r-closeness

Proposition 2. Let μ be an SR measure over $2^{[n]}$, and define ν to be the ESR measure such that $\mu(S) = \alpha\mu(S)^p$ for a given $\alpha \in \mathbb{R}$. Then

$$r(\mu, \nu) \leq \max_{S \in \text{supp}(\nu)} \left[\mu(S)^{-|p-1|}\right] < \infty.$$

Proof. Let μ, ν be as in the proposition statement, and consider $S \in \text{supp}(\nu)$: $\nu(S) > 0$. Recall that $\sum_T \nu(T) = 1$.

$$\frac{\nu(S)}{\mu(S)} = \frac{\mu(S)^p}{\mu(S) \sum_T \mu(T)^p}$$

If $p \leq 1$, we have $\sum_T \mu(T)^p \geq 1$ and $\mu(S)^p \geq \mu(S)$, and so

$$(\min_S \mu(S))^{p-1} \leq \frac{\sum_T \mu(T)}{\sum_T \mu(T)^p} = \frac{1}{\sum_T \mu(T)^p} \leq \frac{\mu(S)^p}{\mu(S) \sum_T \mu(T)^p} \leq \mu(S)^{p-1} \leq \frac{1}{\min_S \mu(S)^{1-p}}$$

where the left inequality is obtained by noticing that $\frac{a+b}{c+d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right)$.

Similarly, for $p \geq 1$, we have $\sum_T \mu(T)^p \leq 1$ and $\mu(T)^p \leq \mu(T)$, and so

$$\min_S \mu(S)^{p-1} \leq \frac{\mu(S)^p}{\mu(S)} \leq \frac{\mu(S)^p}{\mu(S) \sum_T \mu(T)^p} \leq \frac{\sum_T \mu(T)}{\sum_T \mu(T)^p} \leq \max \mu(S)^{1-p} = \frac{1}{\min \mu(S)^{p-1}}.$$

□

C Bounds on mixing times

Theorem 2. Let μ, ν be measures over $2^{[n]}$ such that μ is SR and ν is ESR. Sampling from ν via Alg. 1 with μ as a proposal distribution has a mixing time $\tau(\epsilon)$ such that

$$\tau_S(\epsilon) \leq 2r(\mu, \nu^p) \log \frac{1}{\epsilon}.$$

Proof. Alg. 1 has a state-independent proposal distribution μ, and hence its mixing time is governed by a ratio of probabilities: Cai [14] showed that, after t iterations,

$$\max_{S,T} d_{\text{TV}}(\nu_{(t)}(\cdot \mid S), \nu_{(t)}(\cdot \mid T)) = \left(1 - \frac{1}{\max_U \nu(U)/\mu(U)}\right)^t$$

where $\nu_{(t)}(U \mid S)$ is the probability of being in state U after t iterations when starting from set S, and d_{TV} is the total variation distance.

Hence, following [14, Cor.1], we obtain $\tau_S(\epsilon) \leq 2 \max_U \frac{\nu(U)}{\mu(U)} \log \frac{1}{\epsilon} \leq 2r(\mu, \nu^p) \log \frac{1}{\epsilon}$. □

Theorem 3. Let ν be a k-homogeneous ESR measure over $2^{[n]}$. The mixing time for Alg. 2 with initialization S is bounded in expectation by

$$\tau_S(\epsilon) \leq \inf_{\mu \in \text{SR}} \; 2nk\, r(\mu,\nu)^2 \log \tfrac{1}{\epsilon \nu(S)}$$

Proof. This bound is based on a comparison method [17], and relates the mixing time to the spectral gap. Let $1 = \mu_1 \geq \mu_2 \geq \ldots \geq -1$ be the eigenvalues of the state transition matrix of the chain. The *spectral gap* is $\gamma = 1 - \max\{|\mu|; \mu \text{ is an eigenvalue and } \mu \neq 1\}$. γ directly translates into a bound on the mixing time [18]:

$$\tau_S(\gamma) \leq \frac{1}{\gamma} \log\left(\tfrac{1}{\epsilon \nu(S)}\right).$$

The comparison method yields a bound on γ if we know a bound on $\tilde{\gamma}$ for a related chain with stationary distribution μ. Specifying [17, Thm 2.1] to this case yields $\gamma \geq \tilde{\gamma}\alpha_1/\alpha_2$, where

$$\alpha_1 = \min_S \frac{\mu(S)}{\nu(S)} \geq \frac{1}{r(p)}$$
$$\alpha_2 = \max_{T,U} \frac{\mu(T)}{\nu(T)} \frac{\min\{1, \mu(U)/\mu(T)\}}{\min\{1, \nu(U)/\nu(T)\}} \leq \max_T \frac{\mu(T)}{\nu(T)} \leq r(\mu,\nu)$$

Anari et al. [5] show that $\tilde{\gamma} \geq \frac{1}{2nk}$. Hence, we obtain $\tau_S(\gamma) \leq 2nk \cdot r(\mu,\nu)^2 \cdot \log \frac{1}{\epsilon\nu(S)}$. □

D Bounds for E-DPPs with L^p-kernel proposal

We require the following power-mean inequality:

Theorem 8 (Specht [45])**.** *Let $x_i > 0$ and $w_i \geq 0$ for $1 \leq i \leq N$ such that $\sum_i w_i = 1$. Let $p < q \in \mathbb{R}$ such that $pq \neq 0$. Then, letting $\kappa = \frac{\max x_i}{\min x_i}$,*

$$1 \leq \frac{M_q(\boldsymbol{w};\boldsymbol{x})}{M_p(\boldsymbol{w};\boldsymbol{x})} \leq \left(\frac{q-p}{q}\frac{\kappa^q - 1}{\kappa^q - \kappa^p}\right)^{\frac{1}{p}} \left(\frac{p}{q-p}\frac{\kappa^q - \kappa^p}{\kappa^p - 1}\right)^{\frac{1}{q}}.$$

where the power mean $M_p(\boldsymbol{w};\boldsymbol{x})$ *is defined as*

$$M_p(\boldsymbol{w};\boldsymbol{x}) := \left(\sum\nolimits_{i=1}^N w_i x_i^p\right)^{\frac{1}{p}}.$$

Theorem 9. *Let $L \in \mathbb{P}^n$ be a positive definite matrix and $S \subseteq [n]$. Then,*

$$\det(L[S])^p \geq \det(L^p[S]), \qquad 0 \leq p \leq 1,$$
$$\det(L[S])^p \leq \det(L^p[S]), \qquad p \geq 1.$$

Proof. From Lemma 1, there exists a vector $\boldsymbol{w}$ in the probability simplex, of size $\binom{n}{|S|}$, such that

$$\det(L[S]) = \sum\nolimits_{J \subseteq [n], |J| = |S|} w_J \prod\nolimits_{i \in J} \lambda_i.$$

Since $t \to t^p$ is convex for $p \geq 1$, Jensen's inequality shows that

$$\det(L[S])^p \leq \sum\nolimits_{J \subseteq [n], |J| = |S|} w_J \prod\nolimits_{i \in J} \lambda_i^p = \det(L^p[S]),$$

where the latter equality follows due to L and L^p sharing the same eigenbasis. The same reasoning for $p < 1$ gives the other side of the inequality. □

Theorem 7. Let μ be the distribution induced by a DPP with kernel L^p, and ν be the corresponding E-DPP such that $\nu(S) \triangleq \det(L[S])^p / Z_p$. Then $r(\mu,\nu) \leq r(\kappa_{\lfloor n/2 \rfloor}, p)$ where $r(\kappa, p)$ is defined by

$$r(\kappa,p) = \begin{cases} \left(\frac{p(\kappa-1)}{\kappa^p - 1}\right)^p \left(\frac{(1-p)(\kappa-1)}{\kappa - \kappa^p}\right)^{1-p} & \text{for } 0 < p < 1 \\ \left(\frac{\kappa^p - 1}{p(\kappa-1)}\right)^p \left(\frac{(p-1)(\kappa-1)}{\kappa^p - \kappa}\right)^{p-1} & \text{for } p > 1 \end{cases}$$

Proof. We show the result for general DPPs; the result for k-DPPs follows the same exact reasoning. For $0 < p < 1$, it follows from Thm. 9 that $\det(L[S])^p \geq \det(L^p[S])$. Hence,

$$Z_p = \sum_{S \subseteq [n]} \det(L[S])^p \geq \sum_{S \subseteq [n]} \det(L^p[S]) = \det(I + L^p),$$

which entails $\frac{\det(I+L^p)}{Z_p} \leq 1$ whereby it remains to bound $\frac{\det(L[S])^p}{\det(L^p[S])}$.

Let $S \subseteq [n]$ of size k, and let λ be the vector of L's eigenvalues. We write $\lambda^S = \prod_{i \in S} \lambda_i$, and denote by $\boldsymbol{\lambda}^{\wedge k}$ the $\binom{n}{k}$-vector $(\lambda^S)_{S \subseteq [n], |S|=k}$. Using Lemma 1, there exists $\boldsymbol{w} \in \mathbb{R}^{\binom{n}{k}}$ that sums to 1 such that

$$\begin{aligned}\frac{\det(L[S])^p}{\det(L^p[S])} = \frac{(\sum_{|S|=k} w_S \lambda^S)^p}{\sum_{|S|=k} w_S (\lambda^p)^S} &= \left(\frac{M_1(\boldsymbol{w}; \boldsymbol{\lambda}^{\wedge k})}{M_p(\boldsymbol{w}; \boldsymbol{\lambda}^{\wedge k})}\right)^p \\ &\leq r(p)^p.\end{aligned}$$

Where the last inequality follows from Thm. 8. To lower bound $r_p(S)$, the same reasoning gives us $\frac{\det(L[S]^p)}{\det(L^p[S])} \geq 1$ and hence

$$\begin{aligned}r_p(S) \geq \frac{\det(I + L^p)}{Z_p} &\geq \min_S \frac{\det L^p[S]}{(\det L[S])^p} \\ &\geq \left(\frac{M_p(\boldsymbol{w'}; \boldsymbol{\lambda}^{\wedge k})}{M_1(\boldsymbol{w'}; \boldsymbol{\lambda}^{\wedge k})}\right)^p \geq r(p)^{-p},\end{aligned}$$

where the second inequality follows $\frac{a+b}{c+d} \geq \min(\frac{a}{c}, \frac{b}{d})$. The same reasoning yields the result for $p > 1$.

Finally, some algebra shows that for fixed p, r is an increasing function of κ, and so $r(\kappa_k, p)$ is upper bounded by $r(\kappa_{\lfloor n/2 \rfloor})$. □

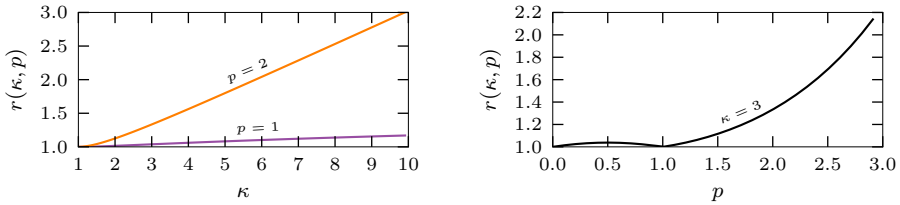
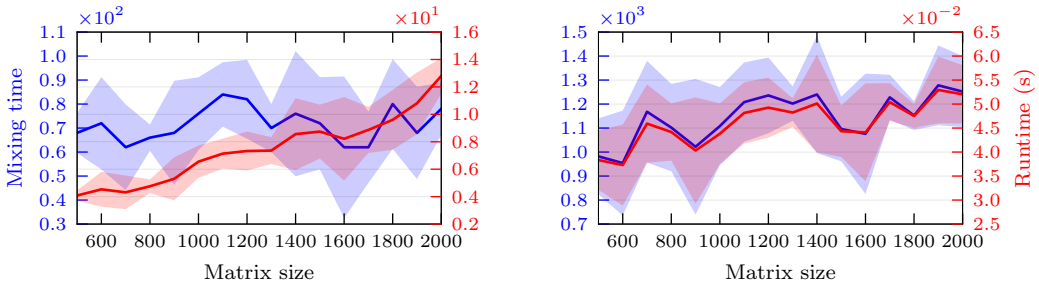


Figure 4: Evolution of the upper bound for $r(p, \kappa)$ from Thm. 7, which measures the r-closeness between the E-DPP with kernel L and the DPP with kernel L^p.

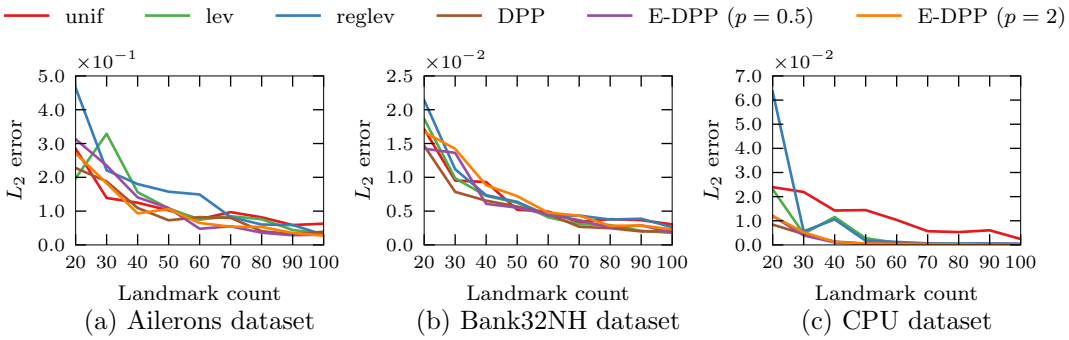
E Mixing time as a function of ground set size for E-DPPs



(a) Proposal, $r = 2.0$ (b) Swapchain, $r = 2.0$

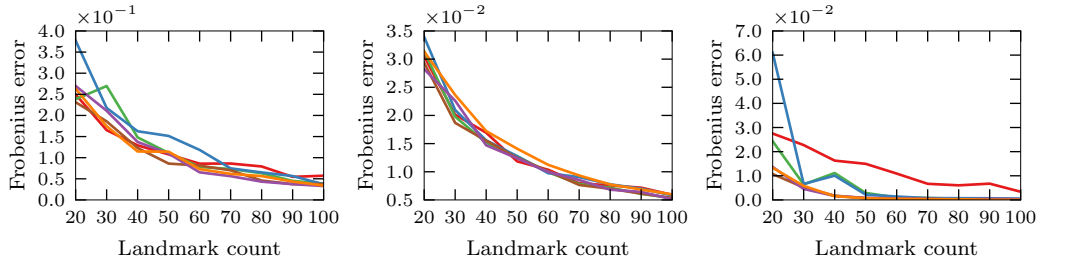
Figure 5: Influence of ground set size n on mixing time

F Additional Nystrom sampling results



(a) Ailerons dataset (b) Bank32NH dataset (c) CPU dataset

Figure 6: L_2 reconstruction error



(a) Ailerons dataset (b) Bank32NH dataset (c) CPU dataset

Figure 7: Frobenius norm reconstruction error