

Supplementary material

Derivation for Lemma 1, 4 As , $r_t^K \leq r_t^0 \leq 2\sqrt{\beta_1}\sigma_t^0$, we can say that

$$\begin{aligned} \sum_{t=1}^T (r_t^K)^2 &\leq \sum 4\beta_1^{uc} (\sigma_t^0)^2 + \sum_{t=1}^T \frac{1}{t^2} \\ &\leq 4\beta_1^{uc} \frac{c_1^{uc} \gamma_T^{uc}}{4} + \frac{\pi^2}{6} \quad (\text{because } \sum_{t=1}^T \frac{1}{t^2} \leq \frac{\pi^2}{6}) \\ &= \beta_1^{uc} c_1^{uc} \gamma_T^{uc} + \frac{\pi^2}{6} \end{aligned}$$

Derivation for A_{T_0} and B_T in Lemma 3 From Equation 7

$$\bar{R}_T \leq 2\sqrt{\frac{\beta_T^c}{T} (A_{T_0} + B_T)} + \frac{1}{T} \sum_{t=1}^T \frac{1}{t^2} \quad (9)$$

For the first term,

$$\begin{aligned} A_{T_0} &= \sum_{t=1}^{T_0} ((\sigma_{t-1}^c(\mathbf{x}_t^c))^2) \\ &= \sum \sigma^2 \overbrace{(\sigma_{r_{T_0}}^{-2} (\sigma_{t-1}^c(x_t^c))^2)}^{s^2} \end{aligned} \quad (10)$$

Now, since $\sigma_{t-1}^c \leq 1$,

$$\begin{aligned} s^2 &= \sigma_{r_{T_0}}^{-2} (\sigma_{t-1}^c(x_t^c))^2 \\ &\leq \sigma_{r_{T_0}}^{-2} \end{aligned}$$

We know if $C_2 = \frac{\sigma_{r_{T_0}}^{-2}}{\log(1+\sigma_{r_{T_0}}^{-2})}$, then $s^2 \leq C_2 \log((1+s^2))$

Thus we have from equation 10,

$$\begin{aligned} A_{T_0} &= \sum \sigma^2 s^2 \\ &\leq \sum \sigma^2 C_2 \log(1 + \sigma_{r_{T_0}}^{-2} (\sigma_{t-1}^c)^2) \\ &= \sum \sigma^2 \frac{\sigma_{r_{T_0}}^{-2}}{\log(1 + \sigma_{r_{T_0}}^{-2})} \log(1 + \sigma_{r_{T_0}}^{-2} (\sigma_{t-1}^c)^2) \\ &= \sum_{t=1}^{T_0} \left(\frac{\log(1 + \sigma_{r_{T_0}}^{-2} (\sigma_{t-1}^c(\mathbf{x}_t^c))^2)}{\log(1 + \sigma_{r_{T_0}}^{-2})} \right) \end{aligned}$$

Since $\sigma_{t-1}^c(\mathbf{x}_t^c) \leq 1$, we have $A_{T_0} \leq \sum_{t=1}^{T_0} \left(\frac{\log(1 + \sigma_{r_{T_0}}^{-2} \times 1)}{\log(1 + \sigma_{r_{T_0}}^{-2})} \right) = T_0$, which is a constant. Hence

$\lim_{T \rightarrow \infty} \frac{\beta_T^c}{T} A_{T_0} \rightarrow 0$.

For the second term,

$$\begin{aligned} B_T &= \sum_{T_0+1}^T ((\sigma_{t-1}^c(\mathbf{x}_t^c))^2) \\ &\leq \sum_{t=T_0+1}^T \left(\frac{\log(1 + (\sigma_{r_{T_0}}^{-2} + \sigma^{-2})(\sigma_{t-1}^c(\mathbf{x}_t^c))^2)}{\log(1 + (\sigma_{r_{T_0}}^{-2} + \sigma^{-2}))} \right) \end{aligned}$$

From Lemma 2 after $T > T_0$ we know $\sigma_{r_{T_0}} < \sigma$, we have

$$\begin{aligned}
B_T &\leq \sum_{t=T_0+1}^T \left(\frac{\log(1 + (\sigma^{-2} + \sigma^{-2})(\sigma_{t-1}^c(\mathbf{x}_t^c)^2))}{\log(1 + (\sigma^{-2} + \sigma^{-2}))} \right) \\
&= \sum_{t=T_0+1}^T \left(\frac{\log(1 + 2\sigma^{-2}(\sigma_{t-1}^c(\mathbf{x}_t^c)^2))}{\log(1 + 2\sigma^{-2})} \right) \\
&= \sum_{t=T_0+1}^T \left(\frac{\log(1 + \tilde{\sigma}^2(\sigma_{t-1}^c(\mathbf{x}_t^c)^2))}{\log(1 + 2\sigma^{-2})} \right) \\
&\leq \frac{1}{\log(1 + 2\sigma^{-2})} \tilde{\gamma}_T
\end{aligned}$$

where $\tilde{\gamma}_T = \max_{A \in \mathcal{X}^c, |A|=T} I(y_A : f_A)$ assuming $y = f + \tilde{\epsilon}$, where $\tilde{\epsilon} \sim \mathcal{N}(0, \sigma^2/2)$. Since $\tilde{\gamma}_T$ grows sublinearly with T , especially for SE kernel $\tilde{\gamma}_T \sim \mathcal{O}((\log T)^{d+1})$, it is easy to show that $\lim_{T \rightarrow \infty} \frac{\beta_T^c}{T} B_T \rightarrow 0$. Hence $\lim_{T \rightarrow \infty} \tilde{R}_T \rightarrow 0$.