

Supplementary Material

A Miscellaneous proofs

A.1 Proof of Corollary 3.2

The proof is standard and uses the self-reducibility of the hard-core model, meaning that conditioning on $\sigma_i = 0$ amounts to removing node i from the graph. Fix a graph G and parameters $\theta = \mathbf{0}$. We show that given an algorithm to approximately compute the marginals for induced subgraphs $H \subseteq G$, it is possible to approximate the partition function $e^{\Phi(\mathbf{0})}$, denoted here by Z . We first claim that

$$Z = \prod_{i=1}^p \frac{1}{1 - \mu_i(G \setminus [i-1])}. \quad (\text{A.1})$$

The graph $G \setminus [i-1]$ is obtained by removing nodes labeled $1, 2, \dots, i-1$, and $\mu_i(G \setminus [i-1])$ is the marginal at node i for this graph. We use induction on the number of nodes. The base case with one node is trivial: $Z = 1 + e^0 = 2 = 1/(1 - \mu)$. Suppose now that the formula (A.1) holds for graphs on k nodes and that $|V| = k+1$. Let Z_0 and Z_1 denote the partition function summation restricted to $\sigma_1 = 0$ or $\sigma_1 = 1$, respectively. Thus

$$Z = Z_0 + Z_1 = Z_0 \left(\frac{Z_0 + Z_1}{Z_0} \right) = \frac{Z_0}{1 - \mu_1}.$$

Now Z_0 is the partition function of a new graph obtained by deleting vertex i , and the inductive assumption proves the formula.

From (A.1) we see that in order to compute a γ -approximation to Z^{-1} , it suffices to compute a γ/p approximation to each of the marginals. Now for small γ , a γ approximation to Z^{-1} gives a 2γ approximation to Z , and this completes the proof.

A.2 Proof of Lemma 4.2

We wish to show that $\mu(\mathbf{0}) \in \mathcal{M}_1$ for a graph $G = (V, E)$ of maximum degree d and $p \geq 2^{d+1}$. Consider a particular node $i \in V$ with neighbors $N(i)$, and let $d_i = |N(i)|$ denote its degree. We use the notation S_i, S_i^-, S_i° defined in Subsection 5.1. A collection of independent set vectors $S \subseteq \mathcal{I}(G)$ is assigned probability $P(S) = |S|/|\mathcal{I}(G)|$ for our choice $\theta = \mathbf{0}$, so it suffices to argue about cardinalities.

We first claim that $|S_i| \geq 2^{-d}|S_i^\circ|$. This follows by observing that each set in S_i° gets mapped to a set in S_i by removing the neighbors N_i , and moreover at most 2^d sets are mapped to the same set in S_i . Next, we note that $|S_i| = |S_i^-|$ since the removal of node i is a bijection from S_i to S_i^- and hence they are of the same cardinality. Combining these observations with the fact that $P(S_i) + P(S_i^-) + P(S_i^\circ) = 1$, we get the estimate $\mu_i = P(S_i) \geq 1/(2^{-d} + 2) \geq 2^{-d-1}$.

Next, we show for each coordinate i that the vector $\mu' = \mu + 2^{-d-1}e_i$ is in \mathcal{M} , which will complete the proof that $\mu(\mathbf{0})$ is \mathcal{M}_1 . Let $\eta_\sigma = P_0(\sigma)$ denote the probability assigned to σ under the distribution with parameters $\theta = \mathbf{0}$, so that $\mu = \sum_{\sigma \in \mathcal{I}(G)} \eta_\sigma \cdot \sigma$. Similarly to the proof of Lemma 5.2, we define a new probability measure

$$\eta'_\sigma = \begin{cases} \eta_\sigma + 2^{-d-1} & \text{if } \sigma \in S_i \\ \eta_\sigma - 2^{-d-1} & \text{if } \sigma \in S_i^- \\ \eta_\sigma & \text{otherwise.} \end{cases}$$

This is a valid probability distribution because $\eta_\sigma \geq 2^{-d-1}$ for $\sigma \in S_i^-$. One can check that $\mu' = \sum_{\sigma \in \mathcal{I}} \eta'_\sigma \sigma$ has $\mu'_j = \mu_j$ for each $j \neq i$ and $\mu'_i = \mu_i + 2^{-d-1}$. The point μ' , being a convex combination of independent set vectors, must be in \mathcal{M} , and hence so must $\mu + 2^{-d-1}e_i$.

B Proofs for projected gradient method

B.1 Proof of Lemma 4.4

The proof here is a slight modification of the proof of Theorem 3.1 in [22].

Observe first that if \mathcal{P} is the projection onto a convex set, then \mathcal{P} is a contraction: $\|\mathcal{P}(x) - \mathcal{P}(y)\|_2 \leq \|x - y\|_2$ (cf. Prop 2.1.3 in [24]). Using the the convexity inequality $G(x) - G(x^*) \leq \nabla G(x)^T(x - x^*)$, the definition $\eta = \sup_{x \in C} \|\widehat{\nabla G}(x) - \nabla G(x)\|_1$, and the update formula $x^{t+1} = x^t - s\widehat{\nabla G}(x^t)$, it follows that

$$\begin{aligned} G(x^t) - G(x^*) &\leq \nabla G(x^t)^T(x^t - x^*) \\ &= \widehat{\nabla G}(x^t)^T(x^t - x^*) + (\widehat{\nabla G}(x^t)^T - \nabla G(x^t)^T)(x^t - x^*) \\ &\leq \widehat{\nabla G}(x^t)^T(x^t - x^*) + \eta\|x^t - x^*\|_\infty \\ &= \frac{1}{s}(x^t - x^{t+1})^T(x^t - x^*) + \eta \\ &= \frac{1}{2s}(\|x^t - x^*\|_2^2 + \|x^t - x^{t+1}\|_2^2 - \|x^{t+1} - x^*\|_2^2) + \eta \\ &= \frac{1}{2s}(\|x^t - x^*\|_2^2 - \|x^{t+1} - x^*\|_2^2) + \frac{s}{2}\|\widehat{\nabla G}(x^t)\|_2^2 + \eta. \end{aligned}$$

Adding the preceding inequality for $t = 1$ to $t = T$, the sum telescopes and we get

$$\sum_{t=1}^T [G(x^t) - G(x^*)] \leq \frac{R^2}{2s} + \frac{s}{2}L^2T + \eta T = RL\sqrt{T} + \eta T. \quad (\text{B.1})$$

Here we used the definitions $R = \|x^1 - x^*\|$ and $L = \sup_{x \in C} \|\widehat{\nabla G}(x)\|$ and the last equality is by the choice $s = \frac{R}{L\sqrt{T}}$. Now defining $\bar{x}^T = \frac{1}{T} \sum_{t=1}^T x^t$, dividing (B.1) through by T and using the convexity of G to apply Jensen's inequality gives

$$G(\bar{x}^T) - G(x^*) \leq \frac{RL}{\sqrt{T}} + \eta.$$

Thus in order to make the right hand side smaller than δ it suffices to take $T = 4R^2L^2/\delta^2$ and $\eta = \delta/2$.

B.2 Proof of Lemma 4.5

We start by showing that the gradient $\nabla \Phi$ is $p^{\frac{3}{2}}$ -Lipschitz. Recall that $\nabla \Phi(\theta) = \mu(\theta)$. We prove a bound on $|\mu_i(\theta) - \mu_i(\theta')|$ by changing one coordinate of θ at a time. Let $\theta^{(r)} = (\theta_1, \dots, \theta_r, \theta'_{r+1}, \dots, \theta'_p)$. The triangle inequality gives

$$|\mu_i(\theta) - \mu_i(\theta')| = \sum_{r=0}^{p-1} |\mu_i(\theta^{(r)}) - \mu_i(\theta^{(r+1)})|.$$

A direct calculation shows that

$$\frac{\partial}{\partial \theta_r} \mu_i(\theta) = \mathbb{P}(\sigma_i = \sigma_r = 1) - \mu_i(\theta) \mu_r(\theta).$$

Since this is uniformly bounded by one in absolute value, we obtain the inequality $|\mu_i(\theta) - \mu_i(\theta')| \leq \|\theta - \theta'\|_1$ or

$$\|\mu(\theta) - \mu(\theta')\|_1 \leq p\|\theta - \theta'\|_1$$

Hence

$$\|\mu(\theta) - \mu(\theta')\|_2 \leq \|\mu(\theta) - \mu(\theta')\|_1 \leq p\|\theta - \theta'\|_1 \leq p^{\frac{3}{2}}\|\theta - \theta'\|_2,$$

i.e., $\nabla \Phi$ is $p^{\frac{3}{2}}$ -Lipschitz.

Now the function $\nabla\Phi$ being $p^{\frac{3}{2}}$ -Lipschitz implies that Φ is $p^{\frac{3}{2}}$ -strongly smooth, where Φ is β -strongly smooth if

$$\Phi(x + \Delta) - \Phi(x) \leq \langle \nabla\Phi(x), \Delta \rangle + \frac{1}{2}\beta\|\Delta\|^2.$$

To see this, we write

$$\begin{aligned} \Phi(x + \Delta) - \Phi(x) &= \int_0^1 \langle \nabla\Phi(x + \tau\Delta), \Delta \rangle d\tau = \langle \nabla\Phi(x), \Delta \rangle + \int_0^1 (\nabla\Phi(x + \tau\Delta) - \nabla\Phi(x)) d\tau \\ &\leq \langle \nabla\Phi(x), \Delta \rangle + p^{\frac{3}{2}} \int_0^1 \langle \tau\Delta, \Delta \rangle d\tau \\ &= \langle \nabla\Phi(x), \Delta \rangle + \frac{1}{2}p^{\frac{3}{2}}\|\Delta\|^2. \end{aligned}$$

Now Theorem 6 from [25] or Chapter 5 of [26] imply that Φ^* , being the Fenchel conjugate of Φ , is $p^{-\frac{3}{2}}$ -strongly convex, meaning

$$\Phi^*(x + \Delta) - \Phi^*(x) \geq \langle \nabla\Phi^*(x), \Delta \rangle + \frac{1}{2}p^{-\frac{3}{2}}\|\Delta\|^2.$$

This gives the desired bound on $\|x - x^*\|$ in terms of $\Phi^*(x) - \Phi^*(x^*)$.

C Proofs of gradient bounds

C.1 Proof of Lemma 5.3

We suppose for the sake of deriving a contradiction that $\theta_i > p/\delta$. Let $\bar{\mu} = \mu + \delta \cdot e_i$, and let η' be a probability measure such that $\bar{\mu} = \sum_{\sigma \in \mathcal{I}} \eta'_\sigma \sigma$. Now $\eta'(S_i) = \bar{\mu}_i \geq \delta$, and we define the non-negative measure γ (summing to less than one) with support S_i as

$$\gamma_\sigma = \begin{cases} \eta'_\sigma \cdot \frac{\delta}{\eta'(S_i)} & \text{if } \sigma \in S_i \\ 0 & \text{otherwise.} \end{cases}$$

In this way, $\gamma_\sigma \leq \eta'_\sigma$ and $\gamma(S_i) = \delta$. We define a new probability measure

$$\eta_\sigma = \begin{cases} \eta'_\sigma - \gamma_\sigma & \text{if } \sigma \in S_i \\ \eta'_\sigma + \gamma_{\sigma \cup \{i\}} & \text{if } \sigma \in S_i^- \\ \eta'_\sigma & \text{otherwise,} \end{cases} \quad (\text{C.1})$$

and one may check that $\mu = \sum_{\sigma \in \mathcal{I}} \eta_\sigma \sigma$ and $\eta(S_i^-) \geq \gamma(S_i) = \delta$. We use the definitions in Subsection 5.1 to get

$$\begin{aligned} F_\mu(\theta) &\triangleq \mu \cdot \theta - \log \left(\sum_{\sigma \in \mathcal{I}} \exp(\sigma \cdot \theta) \right) \\ &= \sum_{\rho \in \mathcal{I}} \eta_\rho \log \frac{\exp(\rho \cdot \theta)}{\sum_{\sigma} \exp(\sigma \cdot \theta)} \\ &\stackrel{(a)}{\leq} \sum_{\rho \in S_i^-} \eta_\rho \log \frac{\exp(\rho \cdot \theta)}{f(S_i^-) + e^{\theta_i} f(S_i^-) + f(S_i^\odot)} \\ &\stackrel{(b)}{\leq} \sum_{\rho \in S_i^-} \eta_\rho \log \frac{f(S_i^-)}{e^{\theta_i} f(S_i^-)} \\ &\leq -\eta(S_i^-) \theta_i \stackrel{(c)}{\leq} -p \stackrel{(d)}{\leq} -\log |\mathcal{I}| = F(\mathbf{0}). \end{aligned}$$

Here (a) follows by restricting the sum to $S_i^- \subseteq \mathcal{I}(G)$ and from the fact that $\sum_{\sigma} \exp(\sigma \cdot \theta) = f(S_i^-) + e^{\theta_i} f(S_i^-) + f(S_i^\odot)$, (b) follows by retaining only the term $e^{\theta_i} f(S_i^-)$ in the denominator and replacing $\exp(\rho \cdot \theta)$ for $\rho \in S_i^-$ with $f(S_i^-) = \sum_{\rho \in S_i^-} \exp(\rho \cdot \theta)$, thereby increasing the argument to the logarithm, (c) uses the fact that $\eta(S_i^-) \geq \delta$ and the assumption that $\theta_i > p/\delta$, and (d) follows from the crude bound on number of independent sets $|\mathcal{I}| \leq 2^p$ and $\log 2 < 1$.

Finally, the relation $\theta(\mu) = \arg \max_{\theta} F_\mu(\theta)$ from Section 3 contradicts $F_\mu(\theta) < F(\mathbf{0})$.

C.2 Proof of Lemma 5.4

We suppose for the sake of contradiction that $\theta_i < -p/\delta$ and show that θ cannot be the vector of canonical parameters corresponding to μ .

Since $\mu \in \mathcal{M}$, there exists a non-negative measure η so that $\mu = \sum_{\sigma \in \mathcal{I}} \eta_\sigma \sigma$, and furthermore $\eta(S_i) = \mu_i \geq \delta$. Now arguments similar to the proof of Lemma 5.3 above give

$$\begin{aligned}
F_\mu(\theta) &= \mu \cdot \theta - \log \left(\sum_{\sigma} \exp(\sigma \cdot \theta) \right) \\
&= \sum_{\rho \in \mathcal{I}} \eta_\rho \log \frac{\exp(\rho \cdot \theta)}{\sum_{\sigma} \exp(\sigma \cdot \theta)} \\
&\leq \sum_{\rho \in S_i} \eta_\rho \log \frac{\exp(\rho \cdot \theta)}{f(S_i^-) + e^{\theta_i} f(S_i^-) + f(S_i^*)} \\
&\leq \sum_{\rho \in S_i} \eta_\rho \log \frac{e^{\theta_i} f(S_i^-)}{f(S_i^-) + e^{\theta_i} f(S_i^-) + f(S_i^*)} \\
&\leq \sum_{\rho \in S_i} \eta_\rho \theta_i = \eta(S_i) \theta_i < -\delta p / \delta = -p \leq -\log |\mathcal{I}| = F(\mathbf{0}).
\end{aligned}$$

As before, this contradicts the relation $\theta(\mu) = \arg \max_{\theta} F_\mu(\theta)$.

D Proof of Proposition 4.6

Starting with x^t in \mathcal{M}_1 , our goal is to show that $x^{t+1} = \mathcal{P}_{\geq}(x^t - s\hat{\theta}(x^t))$ remains in \mathcal{M}_1 . The proof will then follow by induction, because our initial point x^1 is in \mathcal{M}_1 by the hypothesis.

We will use the fact that all hyperplane constraints for \mathcal{M} , except for the non-negativity constraints $x_i \geq 0$, can be written as $\langle h, x \rangle \leq 1$ for a vector $h \in [0, 1]^p$. This can be justified using the fact that $e_i \in \mathcal{M}$ for each i together with the property that for any $\mu \in \mathcal{M}$, any coordinate of μ can be set to zero while remaining in \mathcal{M} .

Given our current iterate x^t , we call a constraint $\langle h, x \rangle \leq 1$ *active* if

$$1 - 2\epsilon \|h\|_{\infty} < \langle h, x^t \rangle \leq 1 - \epsilon \|h\|_{\infty} \quad (\text{D.1})$$

and *critical* if

$$1 - \epsilon \|h\|_{\infty} < \langle h, x^t \rangle. \quad (\text{D.2})$$

Observe that an active constraint has a coordinate i (namely i with $h_i = \|h\|_{\infty}$) with $\langle h, x^t + 2\epsilon \cdot e_i \rangle = \langle h, x^t \rangle + 2h_i\epsilon > 1$ and similarly a critical constraint has a coordinate i with $\langle h, x^t + \epsilon \cdot e_i \rangle = \langle h, x^t \rangle + h_i\epsilon > 1$.

For $x^t \in \mathcal{M}_1$ there are (by definition) no critical constraints, but there may be active constraints. We will first show that inactive constraints can at worst become active for the next iterate x^{t+1} , which requires only that the step-size is not too large relative to the magnitude of the gradient. Then we show that the active constraints have a repulsive property and that x^{t+1} is no closer than x^t to any active constraint, that is, $\langle h, x^{t+1} \rangle \leq \langle h, x^t \rangle$. Thus, if x^t is in \mathcal{M}_1 , then there are no critical constraints for x^{t+1} and every coordinate i satisfies $\langle h, x^{t+1} + \epsilon \cdot e_i \rangle \leq 1$ for all constraint vectors h . Since the projection \mathcal{P}_{\geq} ensures that $x_i^{t+1} \geq q\epsilon$, the update x^{t+1} is in \mathcal{M}_1 . We now focus on inactive constraints.

Inactive constraint. We consider an inactive constraint h , meaning that $\langle h, x^t \rangle + 2\epsilon \|h\|_{\infty} \leq 1$. By assumption the step size $s = (\frac{\epsilon}{2p})^2$ so the increment in any coordinate j is bounded as

$$\begin{aligned}
x_j^{t+1} - x_j^t &\leq s |\hat{\theta}_j(x^t)| \\
&\leq s |\hat{\theta}_j(x^t) - \theta_j(x^t)| + s |\theta_j(x^t)| \\
&\leq (1 + \gamma) s |\theta_j(x^t)| \\
&\leq \epsilon/p
\end{aligned}$$

using Lemma 5.3 and fact that $\gamma \leq 1$. These bounds give

$$\begin{aligned} \langle h, x^{t+1} \rangle &= \langle h, x^t \rangle + \langle h, x^{t+1} - x^t \rangle \leq \langle h, x^t \rangle + \sum_j h_j (x_j^{t+1} - x_j^t) \\ &\leq \langle h, x^t \rangle + p(\epsilon/p) \|h\|_\infty \leq 1 - \epsilon \|h\|_\infty \end{aligned}$$

which shows that the constraint is not critical for x^{t+1} and at worst becomes active.

Active constraint. The rough idea is that if a coordinate i cannot be increased by 2ϵ while remaining in \mathcal{M} , then the parameter θ_i must be sufficiently large, and the next iterate x^{t+1} will decrease enough to overcome the possible increase in other coordinates. This argument does not work, however, because it might be the case that $x_i^t = q\epsilon$, which prevents any decrease (i.e., $x_i^{t+1} \geq x_i^t$) due to the projection \mathcal{P}_\geq . Instead, we start by showing that if *some* coordinate cannot be increased by 2ϵ , then there must be a *reasonably large* coordinate which cannot be increased by $4p\epsilon$.

Lemma D.1. *If h is an active constraint, then there is a coordinate $\ell \in V$ with $x^\ell + (4p\epsilon)e_\ell \notin \mathcal{M}$ and $x_\ell^t \geq 2q\epsilon$.*

Proof. If h is active then $1 - 2\epsilon \|h\|_\infty < \langle h, x^t \rangle$. Using the fact that $h_j \leq 1$ for all j we have

$$1 - 2\epsilon \leq 1 - 2\epsilon \|h\|_\infty < \langle h, x^t \rangle. \quad (\text{D.3})$$

Let $B \subseteq V$ consist of coordinates j with small entries $x_j^t \leq 2\epsilon q$. Then

$$\langle h, x^t \rangle = \sum_{j \in B} h_j x_j^t + \sum_{j \in B^c} h_j x_j^t \leq |B|(2\epsilon q) + \sum_{j \in B^c} h_j x_j^t \leq \frac{2}{p} + \sum_{j \in B^c} h_j x_j^t. \quad (\text{D.4})$$

The last inequality used the crude estimate $|B| \leq p$. Combining (D.3) and (D.4) and rearranging gives

$$\sum_{j \in B^c} h_j x_j^t \geq 1 - 2\epsilon - 2/p \geq 1 - 3/p,$$

and it follows that there is an $\ell \in B^c$ for which $h_\ell \geq h_\ell x_\ell^t \geq 1/2p$. Adding $h_\ell \cdot (4p\epsilon) \geq 2\epsilon$ to both sides of (D.3) shows that $x^\ell + (4p\epsilon)e_\ell$ violates the inequality $\langle h, x \rangle \leq 1$. This proves the lemma, since $x_\ell^t > 2q\epsilon$ for $\ell \in B^c$. \square

We are now ready to prove that $\langle h, x^{t+1} \rangle \leq \langle h, x^t \rangle$. Let ℓ be the coordinate promised by Lemma D.1, with $x^\ell + (4p\epsilon)e_\ell \notin \mathcal{M}$ and $x_\ell^t \geq 2q\epsilon$. From Lemma 5.2, we know that $\theta_\ell(x^t) \geq \log(\frac{q}{4p} - 1) \geq 3 \log p$, for p large enough. By definition of $\hat{\theta}$ being a γ -approximation to θ , $\hat{\theta}_\ell(x^t) \geq (1 - \gamma)\theta_\ell(x^t)$. Therefore, since $\gamma \rightarrow 0$ as $p \rightarrow \infty$, it follows that for p large enough $\hat{\theta}_\ell(x^t) \geq \log p$. This implies

$$x_\ell^{t+1} - x_\ell^t \leq -\min(s\hat{\theta}(x^t), s \log p) \leq -s \log p. \quad (\text{D.5})$$

Here we used the fact that $x_\ell^t \geq q\epsilon + s \log p$ so the projection \mathcal{P}_\geq does not affect this coordinate.

Denote by D the set of coordinates

$$D = \{j \in [p] : \langle h, x^t \rangle + \frac{q}{2}\epsilon h_j > 1\}.$$

These coordinates have non-positive increment: since $x_j \geq q\epsilon$ for $x \in \mathcal{M}_1$, Lemma 5.1 implies that $\theta_j \geq 0$, and hence $\hat{\theta}_j \geq (1 - \gamma)\theta_j \geq 0$, or

$$x_j^{t+1} - x_j^t \leq 0 \quad \text{for } j \in D.$$

In contrast, coordinates in D^c might increase, but by a limited amount: since $x^t \in \mathcal{M}_1$, all coordinates $j \in D^c$ satisfy $x_j^t \geq q\epsilon$, and Lemma 5.4 gives the bound $\theta_j \geq -p/q\epsilon$, or

$$x_j^{t+1} - x_j^t \leq (1 + \gamma)|-s\theta_j| \leq 2sp/q\epsilon \quad \text{for all } j \in D^c. \quad (\text{D.6})$$

Additionally, by the definition of D and the fact that increasing coordinate ℓ by $4p\epsilon$ violates $\langle h, x \rangle \leq 1$, if $j \in D^c$, then $4p\epsilon h_\ell > q\epsilon h_j/2$, or

$$h_j < 8ph_\ell/q \quad \text{for all } j \in D^c. \quad (\text{D.7})$$

Using the crude bound $|D^c| \leq p$ together with (D.6) and (D.7) gives

$$\sum_{j \in D^c} h_j(x_j^{t+1} - x_j^t) \leq |D^c| \frac{8ph_\ell}{q} \cdot \frac{2sp}{q\epsilon} \leq s \frac{4p^2}{q^2\epsilon} h_\ell \leq 4sh_\ell. \quad (\text{D.8})$$

Counting the contributions from D^c in (D.8) in addition to D (none) and ℓ (negative as per (D.5)), it follows that

$$\begin{aligned} \langle c, x^{t+1} \rangle &= \langle h, x^t \rangle + \langle h, x^{t+1} - x^t \rangle \leq \langle h, x^t \rangle + sh_\ell(4 - \theta_\ell) \\ &\leq \langle h, x^t \rangle + sh_\ell(4 - \ln p) \\ &\leq \langle h, x^t \rangle. \end{aligned}$$

Here we have used the fact that p is large enough ($p \geq e^4$ suffices for this last step). In words, we move away from any active hyperplane constraint. This completes the proof.