Supplements to "Characteristic Kernels on Groups and Semigroups"

A   Terminology on groups and semigroups

General references are [2] and [12, Appendix B].

A *semigroup* $(S, \circ)$ is a nonempty set S equipped with an operation $\circ$ that satisfies the associative law;

$$(x \circ y) \circ z = x \circ (y \circ z)$$

for any $x, y, z \in S$.

A semigroup $(S, \circ)$ is said to be *Abelian* if the operation is commutative; *i.e.*,

$$x \circ y = y \circ x$$

for any $x, y \in S$. For an Abelian semigroup, the operation is often denoted by $+$.

Let $(S, \circ)$ be a semigroup. An element $e \in S$ is called a *unit element* if

$$x \circ e = e \circ x = x \qquad (x \in S).$$

The unit element is unique, if it exits.

Suppose that a semigroup $(S, \circ)$ has a unit element e. For $x \in S$, if $y \in S$ satisfies

$$y \circ x = x \circ y = e,$$

y is called the *inverse* of x. The inverse is unique, if it exists, and the inverse of x is denoted by x^{-1}.

A semigroup $(G, \circ)$ is called a *group* if there is a unit element, and every element has its inverse. For an Abelian group, the operation is often denoted by $+$, and the unit element and the inverse of x are written by 0 and $-x$, respectively.

Let S be a semigroup equipped with a topology. S is called a *topological semigroup* if the semigroup operation $(x, y) \mapsto x \circ y$ is continuous with respect to the topology. Likewise, a group G with a topology is called a *topological group* if the group operations $(x, y) \mapsto x \circ y$ and $x \mapsto x^{-1}$ are continuous with respect to the topology.

In general, a topological space X is called *locally compact*, if every $x \in X$ has an open neighborhood W such that the closure of W is compact.

A *locally compact Abelian (LCA) group* is a Hausdorff topological group which is Abelian and locally compact. This class is the topic of Section 3. Examples of LCA groups are $\mathbb{R}$, $\mathbb{T} = [0, 1)$, where the addition is modulo 1, and their direct products $\mathbb{R}^n$ and $\mathbb{T}^n$. Every finite group is also a LCA group with discrete topology.

Typical examples of non-Abelian topological group are the ones consisting of matrices, such as the general linear group $GL(n; \mathbb{K}) = \{A \in M(n \times n; \mathbb{K}) \mid A : \text{invertible}\}$, the special linear group $SL(n) = \{A \in GL(n; \mathbb{K}) \mid det A = 1\}$, the orthogonal group $O(n) = \{A \in GL(n; \mathbb{R}) \mid A^T A = I_n\}$, and the unitary group $U(n) = \{A \in GL(n; \mathbb{C}) \mid A^* A = I_n\}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

B   Terminology on measure theory

General references are [2, Chapter 2] and [12, Appendix E].

Let X be a Hausdorff topological space. A *Radon measure* μ on X is a Borel measure such that
(i) $\mu(K) < \infty$ for every compact set in X,
(ii) (inner regular) $\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ is compact }\}$ for each Borel set E .

The difference between Borel measures and Radon measures is subtle. It is known that on a Polish space, which is a space homeomorphic to a complete separable metric space, any finite Borel measure is automatically a Radon measure.

Let μ_1 and μ_2 be finite measures on a measurable space $(\Omega, \mathcal{B})$. A σ-additive function $\mu_1 - \mu_2$ on $\mathcal{B}$ is called a *signed measure*. We also define a complex-valued measure by $\mu_1 - \mu_2 + \sqrt{-1}(\mu_3 - \mu_4)$.

For a complex-valued Radon measure μ, there is a non-negative Radon measure $|\mu|$ defined by

$$|\mu|(E) = \sup\{\textstyle\sum_i |\mu(E_i)| \mid E = \textstyle\sum_i E_i \text{ is a partition of } E \text{ by Borel sets } E_i\}. \qquad (10)$$

A complex-valued Radon measure μ on X is said to be *regular* if $|\mu|$ is outer regular, that is,

$$|\mu|(E) = \inf\{|\mu|(U) \mid U \text{ is an open set including } E\}$$

for every Borel set E. We define $M(X)$ to be the set of all complex-valued regular measures on X for which $|\mu|(X)$ is finite.

For $\mu \in M(X)$, it is known that there is the largest open set U such that $\mu(U) = 0$. The complement of this open set is called the *support* of μ and denoted by $\mathrm{supp}(\mu)$. By the definition

$$\mathrm{supp}(\mu) = \{x \in X \mid \text{for any open set } U \text{ such that } x \in U, \mu(U) \neq 0\}.$$