Supplementary Material for: An Exponential Lower Bound for Linearly-Realizable MDPs with Constant Suboptimality Gap

1 Proof of Lemma 2

Proof. We first verify the statement for the terminal state f. Observe that at the terminal state f, regardless of the action taken, the next state is always f and the reward is always 0. Hence $Q_h^*(f,\cdot) = V_h^*(f) = 0$ for all $h \in [H]$. Thus $Q_h^*(f,\cdot) = \langle \phi(f,\cdot), v(a^*) \rangle = 0$.

We now verify realizability for other states via induction on $h = H, H-1, \cdots, 1$. The induction hypothesis is $\forall a_1 \in [m]$, $a_2 \neq a_1$,

$$Q_h^*(\overline{a_1}, a_2) = \left(\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right) \cdot \left\langle v(a_2), v(a^*) \right\rangle, \tag{1}$$

and that $\forall a_1 \neq a^*$,

$$V_h^*(\overline{a_1}) = Q_h^*(\overline{a_1}, a^*) = \left\langle v(a_1), v(a^*) \right\rangle + 2\gamma. \tag{2}$$

When $h = H$, (1) holds by the definition of rewards. Next, note that $\forall h$, (2) follows from (1). This is because for $a_2 \neq a^*, a_1$,

$$Q_h^*(\overline{a_1}, a_2) = \left(\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right) \cdot \left\langle v(a_2), v(a^*) \right\rangle \leq 3\gamma^2,$$

while

$$Q_h^*(\overline{a_1}, a^*) = \left\langle v(a_1), v(a^*) \right\rangle + 2\gamma \geq \gamma > 3\gamma^2.$$

In other words, (1) implies that a^* is always the optimal action. Thus, it remains to show that (1) holds for h assuming (2) holds for $h+1$. By Bellman's optimality equation,

$$\begin{aligned} Q_h^*(\overline{a_1}, a_2) &= R_h(\overline{a_1}, a_2) + \mathbb{E}_{s_{h+1}} \left[V_{h+1}^*(s_{h+1}) \middle| \overline{a_1}, a_2 \right] \\ &= -2\gamma \left[\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right] + \Pr[s_{h+1} = \overline{a_2}] \cdot V_{h+1}^*(a_2) + \Pr[s_{h+1} = f] \cdot V_{h+1}^*(f) \\ &= -2\gamma \left[\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right] + \left[\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right] \cdot \left(\left\langle v(a_1), v(a^*) \right\rangle + 2\gamma \right) \\ &= \left(\left\langle v(a_1), v(a_2) \right\rangle + 2\gamma \right) \cdot \left\langle v(a_1), v(a^*) \right\rangle. \end{aligned}$$

This is exactly (1) for h. Hence both (1) and (2) hold for all $h \in [H]$. □

2 Proof of Lemma 5

Proof. We state a proof of this lemma for completeness. By Lemma 4, $\forall s$,

$$\max_{a \in \mathcal{A}} \phi(s,a)^\top \Sigma_s^{-1} \phi(s,a) \leq d.$$

It follows that $\forall a \in \mathcal{A}$,

$$\phi(s,a)\phi(s,a)^\top \preccurlyeq d\Sigma_s.$$

Therefore,

$$\mathbb{E}_{s\sim\nu}\left[\max_{a\in\mathcal{A}}\phi(s,a)^\top\Sigma^{-1}\phi(s,a)\right] = \mathbb{E}_{s\sim\nu}\max_{a\in\mathcal{A}}\mathrm{Tr}\left(\phi(s,a)\phi(s,a)^\top\Sigma^{-1}\right)$$
$$\le \mathbb{E}_{s\sim\nu}\mathrm{Tr}\left(d\Sigma_s\Sigma^{-1}\right) = d^2.$$

□

3 Addressing Footnote 3

Let us redefine $\mathcal{M}_{a^*}$ as follows. The state space is again $\{\bar{1},\cdots,\bar{m},f\}$. The action space is $[m]$ for every state. We will also use the same set of m d-dimensional vectors $\{v_1,\cdots,v_m\}$. In this construction, we will reset $\gamma := \frac{1}{6}$.

Features. The feature map now maps state-action pairs to $d+1$ dimensional vectors, and is defined as follows.

$$\phi(\overline{a_1},a_2) := \left(0,\left(\left\langle v(a_1),v(a_2)\right\rangle + 2\gamma\right)\cdot v(a_2)\right), \qquad (\forall a_1,a_2\in[m], a_1\neq a_2)$$
$$\phi(\overline{a_1},a_1) := \left(\frac{3}{4}\gamma, 0\right), \qquad (\forall a_1\in[m])$$
$$\phi(f,1) = (0,\mathbf{0}),$$
$$\phi(f,a) := (-1,\mathbf{0}). \qquad (\forall a\neq 1)$$

Note that the feature map is again independent of a^*. Define $\theta^* := (1, v(a^*))$.

Rewards. For $1\le h<H$, the rewards are defined as

$$R_h(\overline{a_1},a^*) := \left\langle v(a_1),v(a^*)\right\rangle + 2\gamma, \qquad (a_1\neq a^*)$$
$$R_h(\overline{a_1},a_2) := -2\gamma\left[\left\langle v(a_1),v(a_2)\right\rangle + 2\gamma\right], \qquad (a_2\neq a^*, a_2\neq a_1)$$
$$R_h(\overline{a_1},a_1) := \frac{3}{4}\gamma, \qquad (\forall a_1)$$
$$R_h(f,1) := 0,$$
$$R_h(f,a) := -1. \qquad (a\neq 1)$$

For $h=H$, $r_H(s,a) := \langle\phi(s,a),v(a^*)\rangle$ for every state-action pair.

Transitions. The initial state distribution is set as a uniform distribution over $\{\bar{1},\cdots,\bar{m}\}$. The transition probabilities are set as follows.

$$\Pr[f|\overline{a_1},a^*] = 1,$$
$$\Pr[f|\overline{a_1},a_1] = 1,$$
$$\Pr[\cdot|\overline{a_1},a_2] = \begin{cases}\overline{a_2}: \left\langle v(a_1),v(a_2)\right\rangle + 2\gamma \\ f: 1-\left\langle v(a_1),v(a_2)\right\rangle - 2\gamma\end{cases}, \qquad (a_2\neq a^*, a_2\neq a_1)$$
$$\Pr[f|f,\cdot] = 1.$$

We now check realizability in the new MDP. Note that now we want to show $Q_h^*(s,a) = \phi(s,a)^\top\theta^*$, where $\theta^* = (1, v(a^*))$. We claim that $\forall h\in[H]$,

$$V_h^*(\overline{a_1}) = \langle v(a_1),v(a^*)\rangle + 2\gamma, \qquad (a_1\neq a^*)$$
$$Q_h^*(\overline{a_1},a_2) = (\langle v(a_1),v(a_2)\rangle + 2\gamma)\cdot\langle v(a_2),v(a^*)\rangle, \qquad (a_2\neq a_1)$$
$$Q_h^*(\overline{a_1},a_1) = \frac{3}{4}\gamma. \qquad (\forall a_1)$$

To see this, first notice that the expression of Q_h^* implies that the optimal action is a^* for any non-terminal state. Suppose $a_1 \neq a^*$, then for $a_2 \neq a_1, a^*$, $Q_h^*(\overline{a_1}, a_2) \leq 3\gamma^2 < \gamma \leq Q_h^*(\overline{a_1}, a^*)$. Moreover,

$$Q_h^*(\overline{a_1}, a_1) = \frac{3}{4}\gamma < \gamma \leq Q_h^*(\overline{a_1}, a^*).$$

Thus, a^* is indeed the optimal action for $\overline{a_1}$ if $a_1 \neq a^*$.

For $\overline{a^*}$, $a_1 \neq a^*$, $Q_h^*(\overline{a^*}, a_1) \leq 3\gamma^2 < \frac{3}{4}\gamma = Q_h^*(\overline{a^*}, a^*)$. Therefore, a^* is the optimal action for all states (besides f).

As for f, it is easy to see that $Q_h^*(f, 1) = 0$, and that $\forall a \neq 1$, $Q_h^*(f, a) = -1$.

What remains is show the statements for all h via induction. Suppose that

$$Q_{h+1}^*(\overline{a_1}, a_2) = (\langle v(a_1), v(a_2)\rangle + 2\gamma) \cdot \langle v(a_2), v(a^*)\rangle. \qquad (a_2 \neq a_1)$$

Then indeed $V_{h+1}^*(\overline{a_1}) = Q_{h+1}^*(\overline{a_1}, a^*) = \langle v(a_1), v(a^*)\rangle + 2\gamma$. It follows that $\forall a_2 \neq a^*$

$$\begin{aligned}
Q_h^*(\overline{a_1}, a_2) &= R_h(\overline{a_1}, a_2) + \mathbb{E}_{s_{h+1}}\left[V_{h+1}^*(s_{h+1}) \middle| \overline{a_1}, a_2\right] \\
&= -2\gamma\left[\left\langle v(a_1), v(a_2)\right\rangle + 2\gamma\right] + \Pr[s_{h+1} = \overline{a_2}] \cdot V_{h+1}^*(a_2) + \Pr[s_{h+1} = f] \cdot V_{h+1}^*(f) \\
&= -2\gamma\left[\left\langle v(a_1), v(a_2)\right\rangle + 2\gamma\right] + \left[\left\langle v(a_1), v(a_2)\right\rangle + 2\gamma\right] \cdot \left(\left\langle v(a_1), v(a^*)\right\rangle + 2\gamma\right) \\
&= \left(\left\langle v(a_1), v(a_2)\right\rangle + 2\gamma\right) \cdot \left\langle v(a_1), v(a^*)\right\rangle.
\end{aligned}$$

Suboptimality Gap. In $\mathcal{M}_{a^*}$, $\forall a_1 \neq a^*, \forall a_2 \neq a^*$, $Q_h^*(\overline{a_1}, a_2) \leq \max\{3\gamma^2, \frac{3}{4}\gamma\}$. Thus

$$\Delta_h(\overline{a_1}, a_2) \geq \gamma - \max\{3\gamma^2, \frac{3}{4}\gamma\} = \frac{1}{24}.$$

For $\overline{a^*}$, $V_h^*(\overline{a^*}) = 1 - \gamma$, while for $a_1 \neq a^*$,

$$Q_h^*(\overline{a^*}, a_1) = (\langle v(a^*), v(a_1) + 2\gamma) \cdot \langle v(a^*), v(a_1)\rangle \leq 3\gamma^2.$$

Thus $\Delta_h^*(\overline{a^*}, a_1) \geq \frac{3}{4}\gamma - 3\gamma^2 = \frac{1}{24}$. As for the terminal state f, the suboptimality gap is obviously 1. Therefore $\Delta_{\min} \geq \frac{1}{24}$ in this new MDP.

Information theoretic arguments. The modifications here do not affect the proof of Theorem 1. Suppose action a_2 is taken at state $\overline{a_1}$. If $a_1 \neq a_2$, then the behavior (transitions and rewards) would be identical to the original MDP. If $a_1 = a_2 \neq a^*$, neither the transition and the rewards depend on a^*. Hence, we can still construct a reference MDP as in the proof of Theorem 1, such that information on a^* can only be gained by: (1) either taking a^*; (2) or reaching $s_H \neq f$.

4 Proof of Theorem 1

Theorem 1. *Consider an arbitrary online RL algorithm that takes the feature mapping $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$ as input. In the online RL setting, there exists an MDP with a feature mapping ϕ satisfying Assumption 1 and Assumption 2 with $\Delta_{\min} = \Omega(1)$, such that the algorithm requires $\min\{2^{\Omega(d)}, 2^{\Omega(H)}\}$ samples to find a policy π with*

$$\mathbb{E}_{s_1 \sim \mu} V^\pi(s_1) \geq \mathbb{E}_{s_1 \sim \mu} V^*(s_1) - 0.05$$

with probability 0.1.

Proof. We consider K episodes of interaction between the algorithm and the MDP $\mathcal{M}_a$. Since each trajectory is a sequence of H states, we define the total number of samples as KH. Denote the state, the action and the reward at episode k and timestep h by s_h^k, a_h^k and r_h^k respectively.

Consider the following reference MDP denoted by $\mathcal{M}_0$. The state space, action space, and features of this MDP are the same as those of the MDP family. The transitions are defined as follows:

$$\Pr[\cdot|\overline{a_1}, a_2] = \begin{cases} \overline{a_2} : \left\langle v(a_1), v(a_2)\right\rangle + 2\gamma \\ f : 1 - \left\langle v(a_1), v(a_2)\right\rangle - 2\gamma \end{cases}, \qquad (\forall a_1, a_2 \text{ s.t. } a_1 \neq a_2)$$

$$\Pr[f|f, \cdot] = 1.$$

The rewards are defined as follows:

$$R_h(\overline{a_1}, a_2) := -2\gamma \left[\left\langle v(a_1), v(a_2)\right\rangle + 2\gamma\right], \qquad (\forall a_1, a_2 \text{ s.t. } a_1 \neq a_2)$$
$$R_h(f, \cdot) := 0.$$

Intuitively, this MDP is very similar to the MDP family, except that the optimal action a^* is removed. More specifically, $\mathcal{M}_0$ is identical to $\mathcal{M}_a$ except when the action a is taken at a non-terminal state, or when an episode ends at a non-terminal state.

More specifically, we claim that for $t < H$, $\forall s_t, a_t$ such that $a_t \neq a$,

$$\Pr_{\mathcal{M}_a}[s_{t+1}|s_t, a_t] = \Pr_{\mathcal{M}_0}[s_{t+1}|s_t, a_t],$$

and that for $t < H$, $\forall s_t, a_t$ such that $a_t \neq a$,

$$r_t^{\mathcal{M}_a}(s_t, a_t) = r_t^{\mathcal{M}_0}(s_t, a_t).$$

Also, $r_H^{\mathcal{M}_a}(s_t, a_t) = r_H^{\mathcal{M}_0}(s_t, a_t)$ if $s_t = f$. It follows that

$$\begin{aligned}&\Pr_{\mathcal{M}_a}\left[s_1^1, a_1^1, r_1^1, \cdots s_h^k, a_h^k, r_h^k \middle| a \notin A_h^k, \forall k' \leq k, s_H^{k'} = f\right] \\ =&\Pr_{\mathcal{M}_0}\left[s_1^1, a_1^1, r_1^1, \cdots s_h^k, a_h^k, r_h^k \middle| a \notin A_h^k, \forall k' \leq k, s_H^{k'} = f\right].\end{aligned}$$

Here A_h^k is a shorthand for $\left\{a_1^1, a_2^1, \cdots, a_H^1, \cdots, a_h^k\right\}$, i.e. all actions taken up to timestep h for episode k. By marginalizing the states and the actions, we get

$$\Pr_{\mathcal{M}_a}\left[a_h^k \middle| a \notin A_h^k, \forall k' \leq k, s_H^{k'} = f\right] = \Pr_{\mathcal{M}_0}\left[a_h^k \middle| a \notin A_h^k, \forall k' \leq k, s_H^{k'} = f\right].$$

It then follows that

$$\Pr_{\mathcal{M}_a}\left[a_h^k = a \middle| a \notin A_h^k, \forall k' \leq k, s_H^{k'} = f\right] = \Pr_{\mathcal{M}_0}\left[a_h^k = a \middle| a \notin A_h^k, \forall k' \leq k, s_H^{k'} = f\right].$$

Next, we prove via induction that

$$\Pr_{\mathcal{M}_a}\left[a \in A_h^k \middle| \forall k' \leq k, s_H^{k'} = f\right] = \Pr_{\mathcal{M}_0}\left[a \in A_h^k \middle| \forall k' \leq k, s_H^{k'} = f\right]. \qquad (3)$$

Suppose that (3) holds up to $(k, h-1)$. Then

$$\begin{aligned}&\Pr_{\mathcal{M}_a}\left[a \in A_h^k \middle| \forall k' \leq k, s_H^{k'} = f\right] \\ =&\Pr_{\mathcal{M}_a}\left[a \notin A_{h-1}^k\right] \Pr_{\mathcal{M}_a}\left[a_h^k = a \middle| a \notin A_{h-1}^k, \forall k' \leq k, s_H^{k'} = f\right] + \Pr_{\mathcal{M}_a}\left[a \in A_{h-1}^k \middle| \forall k' \leq k, s_H^{k'} = f\right] \\ =&\Pr_{\mathcal{M}_0}\left[a \notin A_{h-1}^k\right] \Pr_{\mathcal{M}_0}\left[a_h^k = a \middle| a \notin A_{h-1}^k, \forall k' \leq k, s_H^{k'} = f\right] + \Pr_{\mathcal{M}_0}\left[a \in A_{h-1}^k \middle| \forall k' \leq k, s_H^{k'} = f\right] \\ =&\Pr_{\mathcal{M}_0}\left[a \in A_h^k \middle| \forall k' \leq k, s_H^{k'} = f\right].\end{aligned}$$

That is, (3) holds for h, k as well. By induction, (3) holds for all h, k. Thus,

$$\begin{aligned}\Pr_{\mathcal{M}_a}\left[a \in A_h^k\right] &\leq \Pr_{\mathcal{M}_a}\left[a \in A_h^k \middle| \forall k' \leq k, s_H^{k'} = f\right] + \Pr\left[\exists k' \leq k, s_H^{k'} \neq f\right] \\ &\leq \Pr_{\mathcal{M}_0}\left[a \in A_h^k \middle| \forall k' \leq k, s_H^{k'} = f\right] + k \cdot \left(\frac{3}{4}\right)^H.\end{aligned}$$

Since $|A_h^k| \leq kH$, $\sum_{a \in [m]} \Pr_{\mathcal{M}_0}\left[a \in A_h^k \middle| \forall k' \leq k, s_H^{k'} = f\right] \leq kH$. It follows that there exists $a^* \in [m]$ such that

$$\Pr_{\mathcal{M}_0}\left[a^* \in A_H^K \middle| \forall k' \leq K, s_H^{k'} = f\right] \leq \frac{KH}{m} = KH \cdot e^{-\Theta(d)}.$$

As a result

$$\Pr_{\mathcal{M}_{a^*}}\left[a^* \in A_H^K\right] \leq KH \cdot e^{-\Theta(d)} + K\left(\frac{3}{4}\right)^H.$$

In other words, unless $KH = 2^{\Omega(\min\{d,H\})}$, the probability of taking the optimal action a^* in the interaction with $\mathcal{M}_{a^*}$ is $o(1)$.

From the suboptimality gap condition, it follows that if $\mathbb{E}_{s_1\sim\mu}V^\pi(s_1) \ge \mathbb{E}_{s_1\sim\mu}V^*(s_1) - 0.05$, $\Pr\left[a_1 \neq a^* \wedge s_1 \neq \overline{a^*}\right] \cdot \Delta_{\min} \le 0.05$. Hence

$$\Pr\left[a_1 = a^*\right] \ge 1 - \left(0.8 + \frac{1}{m}\right) = 0.2 - \frac{1}{m}.$$

Therefore, if the algorithm is able to output such a policy with probability 0.1, it is able to take the action a^* in the next episode with $\Theta(1)$ probability by executing π. However, as proved above, this is impossible unless $KH = 2^{\Omega(\min\{d,H\})}$. □

5 Proof of Theorem 2

Recall the statements of Assumptions 3 and 4.

Assumption 3 (Low variance condition)**.** There exists a constant $1 \le C_{\text{var}} < \infty$ such that for any $h \in [H]$ and any policy π,

$$\mathbb{E}_{s\sim\mathcal{D}_h^\pi}\left[\left|V^\pi(s) - V^*(s)\right|^2\right] \le C_{\text{var}} \cdot \left(\mathbb{E}_{s\sim\mathcal{D}_h^\pi}\left[\left|V^\pi(s) - V^*(s)\right|\right]\right)^2.$$

Assumption 4. There exists a constant $1 \le C_{\text{hyper}} < \infty$ such that for any $h \in [H]$ and any policy π, the distribution of $\phi(s,a)$ with $(s,a) \sim \mathcal{D}_h^\pi$ is $(C_{\text{hyper}}, 4)$-hypercontractive. In other words, $\forall \pi$, $\forall h \in [H]$, $\forall v \in \mathbb{R}^d$,

$$\mathbb{E}_{(s,a)\sim\mathcal{D}_h^\pi}\left[(\phi(s,a)^\top v)^4\right] \le C_{\text{hyper}} \cdot \left(\mathbb{E}_{(s,a)\sim\mathcal{D}_h^\pi}[(\phi(s,a)^\top v)^2]\right)^2.$$

Theorem 2. *Assume that Assumption 1, 2, and one of Assumption 3 and 4 hold. Also assume that*

$$\epsilon \le \text{poly}(\Delta_{\min}, 1/C_{\text{var}}, 1/d, 1/H) \qquad \text{(Under Assumption 3)}$$
$$\text{or} \quad \epsilon \le \text{poly}(\Delta_{\min}, 1/C_{\text{hyper}}, 1/d, 1/H). \qquad \text{(Under Assumption 4)}$$

Let μ be the initial state distribution. Then with probability $1-\epsilon$, running Algorithm 1 on input 0 returns a policy π which satisfies $\mathbb{E}_{s_1\sim\mu}V^\pi(s_1) \ge \mathbb{E}_{s_1\sim\mu}V^*(s_1) - \epsilon$ using $\text{poly}(1/\epsilon)$ trajectories.

Proof under Assumption 3. Let us set $\beta = 8$, $\lambda_{\text{ridge}} = \epsilon^2$, $\lambda_r = \epsilon^6$, $B = 2d\log(\frac{d}{\lambda_r})$, $\epsilon_1 = \epsilon^2$, $\epsilon_2 = \frac{\lambda_r}{2B}$, $N = \frac{d\cdot\log(1/\epsilon_2)}{\epsilon_2^2}$. Recall that $\epsilon \le \text{poly}(\Delta_{\min}, 1/C_{\text{var}}, 1/d, 1/H)$. First, by Lemma 8, the event Ω holds with probability $1-\epsilon$; we will condition on this event in the following proof. By lemma 10, when the algorithm terminates, $|\Pi_h| \le B$ for all $h \in [H]$. Note that the this implies that Algorithm 1 is called or restarted at most $H \cdot (1+B)$ times. In each call or restart of Algorithm 1, at most $NB + N$ trajectories are sampled. Therefore, when the algorithm terminates, at most

$$H(1+B)\cdot(NB+N) \le \text{poly}\left(1/\epsilon\right)$$

trajectories are sampled.

It remains to show that the greedy policy with respect to $\theta_1, \cdots, \theta_H$ is indeed ϵ-optimal with high probability. To that end, let us state the following claims about the algorithm.

1. Each time Line 9 is reached in Algorithm 1, $\forall \pi \in \Pi_h$, define $\tilde{\pi}_h$ as in (6), $\forall h' > h$,

$$\mathbb{E}_{s_{h'}\sim\mathcal{D}_{h'}^{\tilde{\pi}_h}}\left[\sup_{a\in\mathcal{A}}\left|\phi(s_{h'},a)^\top(\theta_{h'} - \theta_{h'}^*)\right|^2\right] \le \frac{\Delta_{\min}^2\epsilon}{4H}. \tag{4}$$

2. Each time when θ_h is updated at Line 17, $\forall \pi \in \Pi_h$, define the associated covariance matrix at step h as $\Sigma_h^\pi = \mathbb{E}_{s_h\sim\mathcal{D}_h^\pi, a_h\sim\rho_{s_h}}\left[\phi(s_h,a_h)\phi(s_h,a_h)^\top\right]$. Then $\|\theta_h - \theta_h^*\|_{\Sigma_h^\pi}^2 \le 6BC_{\text{var}}\epsilon^2$. It follows that

$$\mathbb{E}_{s_h\sim\mathcal{D}_h^\pi}\left[\sup_{a\in\mathcal{A}}\left|\phi(s_h,a)^\top(\theta_h - \theta_h^*)\right|^2\right] \le \frac{\Delta_{\min}^2\epsilon}{4H}. \tag{5}$$

Note that by the first claim with $h=0$, it follows that for the greedy policy $\hat{\pi}$ ($\tilde{\pi}_0$ is always the greedy policy) w.r.t. $\{\theta_h\}_{h\in[H]}$, $\forall h \in [H]$,

$$\mathbb{E}_{s_h\sim\mathcal{D}_h^{\hat{\pi}}}\left[\sup_{a\in\mathcal{A}}\left|\phi(s_h,a)^\top(\theta_h-\theta_h^*)\right|^2\right]\le\frac{\Delta_{\min}^2\epsilon}{4H}.$$

Consequently by Markov's inequality,

$$\Pr_{s_h\sim\mathcal{D}_h^{\hat{\pi}}}\left[\exists a\in\mathcal{A}:\left|\phi(s_h,a)^\top(\theta_h-\theta_h^*)\right|>\frac{\Delta_{\min}}{2}\right]\le\frac{\epsilon}{H}.$$

By Assumption 2 and the fact that $\hat{\pi}$ takes the greedy action w.r.t. θ_h, this implies that

$$\Pr_{s_h\sim\mathcal{D}_h^{\hat{\pi}}}\left[\hat{\pi}_h(s_h)\ne\pi_h^*(s_h)\right]\le\frac{\epsilon}{H}.$$

Thus for a random trajectory induced by $\hat{\pi}$, with probability at least $1-\epsilon$, $\hat{\pi}_h(s_h)=\pi_h^*(s_h)$ for all $h=1,\cdots,H$, which proves the theorem.

It remains to prove the two claims.

Proof of (5). We first prove the second claim based on the assumption that the first claim holds when Line 9 is reached in the same execution of LearnLevel. By the first claim and the same arguments above, $\forall\pi\in\Pi_h$, construct $\tilde{\pi}_h$ as

$$\tilde{\pi}_h(s_{h'})=\begin{cases}\pi(s_{h'}) & (\text{if } h'<h)\\ \text{Sample from } \rho_{s_h}(\cdot) & (\text{if } h'=h)\\ \arg\max_a \phi_{h'}(s_{h'},a)^\top\theta_{h'} & (\text{if } h'>h)\end{cases}, \tag{6}$$

then $\Pr_{s_{h'}\sim\mathcal{D}_{h'}^{\tilde{\pi}_h}}\left[\tilde{\pi}_h(s_{h'})\ne\pi^*(s_{h'})\right]\le\epsilon/H$. Thus,

$$\mathbb{E}_{s_{h+1}\sim\mathcal{D}_{h+1}^{\tilde{\pi}_h}}\left[V_{h+1}^{\tilde{\pi}_h}(s_{h+1})\right]\ge\mathbb{E}_{s_{h+1}\sim\mathcal{D}_{h+1}^{\tilde{\pi}_h}}\left[V_{h+1}^{*}(s_{h+1})\right]-\epsilon.$$

By Assumption 3, this suggests that

$$\mathbb{E}_{s_{h+1}\sim\mathcal{D}_{h+1}^{\tilde{\pi}_h}}\left[\left(V_{h+1}^{\tilde{\pi}_h}(s_{h+1})-V_{h+1}^{*}(s_{h+1})\right)^2\right]\le C_{\text{var}}\epsilon^2.$$

When (s_h,a_h,y) is sampled,

$$\begin{aligned}\mathbb{E}\left[y|s_h,a_h\right]&=\mathbb{E}\left[R(s_h,a_h)+V_{h+1}^{\tilde{\pi}_h}(s_{h+1})|s_h,a_h\right]\\&=Q^*(s_h,a_h)+\mathbb{E}\left[V_{h+1}^{\tilde{\pi}_h}(s_{h+1})-V_{h+1}^{*}(s_{h+1})|s_h,a_h\right],\end{aligned}$$

where the expectation is over trajectories induced by $\tilde{\pi}_h$. In other words, $y_i:=\sum_{h'\ge h}r_h^i$ can be written as $\phi(s_h^i,a_h^i)^\top\theta_h^*+b_i+\xi_i$, where ξ_i is mean-zero independent noise with $|\xi_i|\le 2$ almost surely and $b_i:=\sum_{h'>h}r_{h'}^i-V_{h+1}^*(s_{h+1}^i)$ satisfies $\mathbb{E}[b_i^2]\le C_{\text{var}}\epsilon^2$. Note that θ_h is the ridge regression estimator for this linear model. By Lemma 7,

$$\mathbb{E}_{\pi\sim\text{Unif}(\Pi_h),s_h\sim\mathcal{D}_h^\pi,a_h\sim\rho_{s_h}}\left[\left|\phi(s_h,a_h)^\top(\theta_h-\theta_h^*)\right|^2\right]\le 4(C_{\text{var}}\epsilon^2+\epsilon_1+\lambda_{\text{ridge}})\le 6C_{\text{var}}\epsilon^2.$$

It follows that $\forall\pi\in\Pi_h$,

$$\mathbb{E}_{s_h\sim\mathcal{D}_h^\pi,a_h\sim\rho_{s_h}}\left[\left|\phi(s_h,a_h)^\top(\theta_h-\theta_h^*)\right|^2\right]\le|\Pi_h|\cdot 6C_{\text{var}}\epsilon^2\le 6BC_{\text{var}}\epsilon^2.$$

Now, by Lemma 5,

$$\begin{aligned}&\mathbb{E}_{s_h\sim\mathcal{D}_h^\pi}\left[\sup_{a\in\mathcal{A}}\left|\phi(s_h,a)^\top(\theta_h-\theta_h^*)\right|^2\right]\\\le&\mathbb{E}_{s_h\sim\mathcal{D}_h^\pi}\left[\sup_{a\in\mathcal{A}}\|\phi(s_h,a)\|_{(\Sigma_h^\pi)^{-1}}^2\right]\cdot\|\phi_h-\phi_h^*\|_{\Sigma_h^\pi}^2\\\le&d^2\cdot 6BC_{\text{var}}\epsilon^2\le\frac{\Delta_{\min}^2\epsilon}{4H}.\end{aligned}$$

This proves the second claim.

Proof of (4). Now, let us prove the first claim, assuming that the second claim holds for the last update of any θ_h. By observing Algorithm 1, if Line 9 is reached, during the last execution of the first `for` loop (i.e. Lines 1 to 8), the `if` clause at Line 5 must have returned False every time (otherwise the algorithm will restart). It follows that during the last execution of Lines 1 to 8, neither $\{\theta_h\}_{h\in[H]}$ nor $\{\Pi_h\}_{h\in[H]}$ is updated.

Consider the `if` clause when checking $\pi \in \Pi_h$ for layer h'. Recall that

$$\Sigma_{h'}^{\tilde{\pi}_h} = \mathbb{E}_{s_{h'}\sim\mathcal{D}_{h'}^{\tilde{\pi}_h}, a_{h'}\sim\rho_{s_{h'}}} \left[\phi(s_{h'}, a_{h'})\phi(s_{h'}, a_{h'})^\top\right].$$

Also define $\Sigma_{h'}^* := \frac{\lambda_r}{|\Pi_{h'}|} I + \mathbb{E}_{\pi\sim\text{Unif}(\Pi_{h'})}\Sigma_{h'}^{\pi}$. Then by Lemma 9,

$$\| \left(\Sigma_{h'}^*\right)^{-\frac{1}{2}} \Sigma_{h'}^{\tilde{\pi}_h} \left(\Sigma_{h'}^*\right)^{-\frac{1}{2}} \|_2 \le 3\beta|\Pi_{h'}|.$$

It follows that

$$\begin{aligned}
\|\theta_{h'} - \theta_{h'}^*\|_{\Sigma_{h'}^{\tilde{\pi}_h}}^2 &= (\theta_{h'} - \theta_{h'}^*)^\top \Sigma_{h'}^{\tilde{\pi}_h} (\theta_{h'} - \theta_{h'}^*) \\
&= \left((\Sigma_{h'}^*)^{\frac{1}{2}} (\theta_{h'} - \theta_{h'}^*) \right)^\top \left((\Sigma_{h'}^*)^{-\frac{1}{2}} \Sigma_{h'}^{\tilde{\pi}_h} (\Sigma_{h'}^*)^{-\frac{1}{2}} \right) \left((\Sigma_{h'}^*)^{\frac{1}{2}} (\theta_{h'} - \theta_{h'}^*) \right) \\
&\le \|\theta_{h'} - \theta_{h'}^*\|_{\Sigma_{h'}^*}^2 \cdot \| (\Sigma_{h'}^*)^{-\frac{1}{2}} \Sigma_{h'}^{\tilde{\pi}_h} (\Sigma_{h'}^*)^{-\frac{1}{2}} \|_2 \\
&\le 3\beta B \cdot \left(\lambda_r \cdot \left(\frac{2}{\lambda_{\text{ridge}}} \right)^2 + 6BC_{\text{var}}\epsilon^2 \right) \\
&\le 24B^2 \cdot 10C_{\text{var}}\epsilon^2.
\end{aligned}$$

By Lemma 5,

$$\mathbb{E}_{s_{h'}\sim\mathcal{D}_h^{\tilde{\pi}_h}} \left[\sup_{a\in\mathcal{A}} \|\phi(s_{h'}, a)\|_{(\Sigma_{h'}^{\tilde{\pi}_h})^{-1}}^2 \right] \le d^2.$$

As a result,

$$\begin{aligned}
\mathbb{E}_{s_{h'}\sim\mathcal{D}_{h'}^{\tilde{\pi}_h}} \left[\sup_{a\in\mathcal{A}} \left|\phi(s_{h'}, a)^\top (\theta_{h'} - \theta_{h'}^*)\right|^2 \right] &\le \mathbb{E}_{s_{h'}\sim\mathcal{D}_h^{\tilde{\pi}_h}} \left[\|\theta_{h'} - \theta_{h'}^*\|_{\Sigma_{h'}^{\tilde{\pi}_h}}^2 \cdot \sup_{a\in\mathcal{A}} \|\phi(s_{h'}, a)\|_{(\Sigma_{h'}^{\tilde{\pi}_h})^{-1}}^2 \right] \\
&\le 240B^2C_{\text{var}}\epsilon^2 \cdot d^2 \le \frac{\epsilon\Delta_{\min}^2}{4H}.
\end{aligned}$$

This proves the first claim. The failure probability of the algorithm is controlled by Lemma 8. ☐

Proof under Assumption 4. The proof under Assumption 4 is quite similar, except that we will use Lemma 14 instead of Lemma 7 for the analysis of ridge regression. The different analysis of ridge regression results in a slightly different choice of algorithmic parameters.

Let us set $\beta = 8$, $\epsilon_0 = \epsilon^2$, $\lambda_{\text{ridge}} = \epsilon^3$, $\lambda_r = \epsilon^9$, $B = 2d\log(\frac{d}{\lambda_r})$, $\epsilon_1 = \epsilon^3$, $\epsilon_2 = \frac{\lambda_r}{2B}$, $N = \frac{d}{\epsilon_2^3}$. Recall that $\epsilon \le \text{poly}(\Delta_{\min}, 1/C_{\text{hyper}}, 1/d, 1/H)$. We will state similar claims about the algorithm.

1. Each time Line 9 is reached in Algorithm 1, $\forall\pi \in \Pi_h$, define $\tilde{\pi}_h$ as in (6), $\forall h' > h$,

$$\mathbb{E}_{s_{h'}\sim\mathcal{D}_{h'}^{\tilde{\pi}_h}} \left[\sup_{a\in\mathcal{A}} \left|\phi(s_{h'}, a)^\top (\theta_{h'} - \theta_{h'}^*)\right|^2 \right] \le \frac{\Delta_{\min}^2\epsilon_0}{4H}. \tag{7}$$

2. Each time when θ_h is updated at Line 17, $\forall\pi \in \Pi_h$, define the associated covariance matrix at step h as $\Sigma_h^\pi = \mathbb{E}_{s_h\sim\mathcal{D}_h^\pi, a_h\sim\rho_{s_h}} \left[\phi(s_h, a_h)\phi(s_h, a_h)^\top\right]$. Then $\|\theta_h - \theta_h^*\|_{\Sigma_h^\pi}^2 \le \frac{\Delta_{\min}^2\epsilon_0}{120HBd^2}$. It follows that

$$\mathbb{E}_{s_h\sim\mathcal{D}_h^\pi} \left[\sup_{a\in\mathcal{A}} \left|\phi(s_h, a)^\top (\theta_h - \theta_h^*)\right|^2 \right] \le \frac{\Delta_{\min}^2\epsilon_0}{4H}. \tag{8}$$

As in the proof under Assumption 3, these two claims are sufficient to guarantee that the greedy policy induced by $\{\theta_h\}_{h\in[H]}$ is ϵ-optimal. We now prove the two claims in similar fashion.

Proof of (8). We first prove the second claim based on the assumption that the first claim holds when Line 9 is reached in the same execution of LearnLevel. By the first claim, $\forall \pi \in \Pi_h$, construct $\tilde{\pi}_h$ as in (6), then

$$\Pr_{s_{h'} \sim \mathcal{D}_{h'}^{\tilde{\pi}_h}} \left[\tilde{\pi}_h(s_{h'}) \neq \pi^*(s_{h'})\right] \leq \epsilon_0 / H. \tag{9}$$

When (s_h, a_h, y) is sampled,

$$\begin{aligned}\mathbb{E}\left[y | s_h, a_h\right] &= \mathbb{E}\left[R(s_h, a_h) + V_{h+1}^{\tilde{\pi}_h}(s_{h+1}) | s_h, a_h\right] \\ &= Q^*(s_h, a_h) + \mathbb{E}\left[V_{h+1}^{\tilde{\pi}_h}(s_{h+1}) - V_{h+1}^*(s_{h+1}) | s_h, a_h\right],\end{aligned}$$

where the expectation is over trajectories induced by $\tilde{\pi}_h$. In other words, $y_i := \sum_{h' \geq h} r_h^i$ can be written as $\phi(s_h^i, a_h^i)^\top \theta_h^* + b_i + \xi_i$, where ξ_i is mean-zero independent noise with $|\xi_i| \leq 2$ almost surely, and b_i is defined as

$$b_i := -\sum_{h' > h} \left(V^*(s_{h'}^i) - Q^*(s_{h'}^i, a_{h'}^i)\right).$$

Here $\mathbb{E}[\xi_i] = 0$ because

$$\mathbb{E}[\xi_i] = \mathbb{E}\left[\sum_{h' \geq h} r_{h'}^i\right] - Q_h^*(s_h^i, a_h^i) - \mathbb{E}[b_i] = Q^{\tilde{\pi}_h}(s_h^i, a_h^i) - Q^*(s_h^i, a_h^i) + \left(Q^*(s_h^i, a_h^i) - Q^{\tilde{\pi}_h}(s_h^i, a_h^i)\right) = 0.$$

By (9), $\Pr[b_i \neq 0] \leq \epsilon_0$. Thus by Lemma 14,

$$\begin{aligned}\mathbb{E}_{\pi \sim \text{Unif}(\Pi_h), s_h \sim \mathcal{D}_h^\pi, a_h \sim \rho_{s_h}}\left[\left|\phi(s_h, a_h)^\top (\theta_h - \theta_h^*)\right|^2\right] &\leq 8\left(\epsilon_1 + \lambda_{\text{ridge}}\right) + 288 \epsilon_0^{1.5} C_{\text{hyper}}^{2.5} d^{4.5} \left(\frac{2B}{\epsilon}\right)^{0.5} \\ &\leq 16\epsilon^3 + 288 \epsilon^{2.5} C_{\text{hyper}}^{2.5} d^{4.5} (2B)^{0.5}.\end{aligned}$$

It follows that $\forall \pi \in \Pi_h$,

$$\mathbb{E}_{s_h \sim \mathcal{D}_h^\pi, a_h \sim \rho_{s_h}}\left[\left|\phi(s_h, a_h)^\top (\theta_h - \theta_h^*)\right|^2\right] \leq |\Pi_h| \cdot \left(16\epsilon^2 + 288\epsilon^{2.5} C_{\text{hyper}}^{2.5} d^{4.5} (2B)^{0.5}\right) \leq \frac{\Delta_{\min}^2 \epsilon_0}{120 H B d^2},$$

where we used the fact $\epsilon \leq \text{poly}(\Delta_{\min}, 1/C_{\text{hyper}}, 1/d, 1/H)$. Now, by Lemma 5,

$$\begin{aligned}\mathbb{E}_{s_h \sim \mathcal{D}_h^\pi}\left[\sup_{a \in \mathcal{A}} \left|\phi(s_h, a)^\top (\theta_h - \theta_h^*)\right|^2\right] \leq& \mathbb{E}_{s_h \sim \mathcal{D}_h^\pi}\left[\sup_{a \in \mathcal{A}} \|\phi(s_h, a)\|_{(\Sigma_h^\pi)^{-1}}^2\right] \cdot \|\phi_h - \phi_h^*\|_{\Sigma_h^\pi}^2 \\ \leq& d^2 \cdot \frac{\Delta_{\min}^2 \epsilon_0}{120 H B d^2} \leq \frac{\Delta_{\min}^2 \epsilon_0}{4H}.\end{aligned}$$

This proves the second claim.

Proof of (7). Now, let us prove the first claim, assuming that the second claim holds for the last update of any θ_h. Consider Line 9 when checking for $\pi \in \Pi_h$ for layer h'. Recall that

$$\Sigma_{h'}^{\tilde{\pi}_h} = \mathbb{E}_{s_{h'} \sim \mathcal{D}_h^{\tilde{\pi}_h}, a_{h'} \sim \rho_{s_{h'}}}\left[\phi(s_{h'}, a_{h'}) \phi(s_{h'}, a_{h'})^\top\right].$$

Similar to the proof under Assumption 3, we can bound $\|\theta_{h'} - \theta_{h'}^*\|_{\Sigma_{h'}^{\tilde{\pi}_h}}$ by

$$\begin{aligned}\|\theta_{h'} - \theta_{h'}^*\|_{\Sigma_{h'}^{\tilde{\pi}_h}}^2 &\leq \|\theta_{h'} - \theta_{h'}^*\|_{\Sigma_{h'}^*}^2 \cdot \|\left(\Sigma_{h'}^*\right)^{-\frac{1}{2}} \Sigma_{h'}^{\tilde{\pi}_h} \left(\Sigma_{h'}^*\right)^{-\frac{1}{2}}\|_2 \\ &\leq 3\beta B \cdot \left(\lambda_r \cdot \left(\frac{2}{\lambda_{\text{ridge}}}\right)^2 + \frac{\Delta_{\min}^2 \epsilon_0}{120 H B d^2}\right) \\ &\leq 96 B \epsilon^3 + \frac{\Delta_{\min} \epsilon_0}{5 H d^2}.\end{aligned}$$

By Lemma 5, $\mathbb{E}_{s_{h'}\sim\mathcal{D}_h^{\bar{\pi}_h}}\left[\sup_{a\in\mathcal{A}}\|\phi(s_{h'},a)\|^2_{(\Sigma_{h'}^{\bar{\pi}_h})^{-1}}\right]\le d^2$. Consequently

$$\mathbb{E}_{s_{h'}\sim\mathcal{D}_{h'}^{\bar{\pi}_h}}\left[\sup_{a\in\mathcal{A}}\left|\phi(s_{h'},a)^\top(\theta_{h'}-\theta^*_{h'})\right|^2\right]\le \mathbb{E}_{s_{h'}\sim\mathcal{D}_h^{\bar{\pi}_h}}\left[\|\theta_{h'}-\theta^*_{h'}\|^2_{\Sigma_{h'}^{\bar{\pi}_h}}\cdot\sup_{a\in\mathcal{A}}\|\phi(s_{h'},a)\|^2_{(\Sigma_{h'}^{\bar{\pi}_h})^{-1}}\right]$$
$$\le 96B\epsilon^3d^2+\frac{\Delta_{\min}\epsilon_0}{5H}\le\frac{\Delta_{\min}\epsilon_0}{4H}.$$

In the last inequality we used $\epsilon_0=\epsilon^2$ and $\epsilon\le\mathrm{poly}(\Delta_{\min},1/d,1/H)$. This proves (7). Finally the failure probability is controlled in Lemma 8. □

Lemma 6 (Covariance concentration [Tropp, 2015]). *Suppose $M_1,\cdots,M_N\in\mathbb{R}^{d\times d}$ are i.i.d. random matrices drawn from a distribution $\mathcal{D}$ over positive semi-definite matrices. If $\|M_t\|_F\le 1$ almost surely and $N=\Omega\left(\frac{d\log(d/\delta)}{\epsilon^2}\right)$, then with probability $1-\delta$,*

$$\left\|\frac{1}{N}\sum_{i=1}^N M_t-\mathbb{E}_{M\sim\mathcal{D}}[M]\right\|_2\le\epsilon.$$

Lemma 7 (Risk bound for ridge regression, Lemma A.2 Du et al. [2019]). *Suppose that (x_1,y_1), $\cdots$, (x_N,y_N) are i.i.d. data drawn from $\mathcal{D}$ with*

$$y_i=\theta^\top x_i+b_i+\xi_i,$$

where $\mathbb{E}_{(x_i,y_i)\sim\mathcal{D}}[b_i^2]\le\eta$, $|\xi_i|\le 2n$ almost surely and $\mathbb{E}[\xi_i]=0$. Let the ridge regression estimator be

$$\hat{\theta}=\left(\sum_{i=1}^N x_ix_i^\top+N\lambda_{\mathrm{ridge}}\cdot I\right)^{-1}\cdot\sum_{i=1}^N x_iy_i.$$

If $N=\Omega\left(\frac{d}{\epsilon_N^2}\log(\frac{d}{\delta})\right)$, then with probability at least $1-\delta$,

$$\mathbb{E}_{x\sim\mathcal{D}}\left[\left((\hat{\theta}-\theta)^\top x\right)^2\right]\le 4\left(\eta+\epsilon_N+\lambda_{\mathrm{ridge}}\right).$$

Lemma 8 (Failure probability). *Define the following events regarding the execution of Algorithm 1.*

1. *Ω_1: Each time Σ_h is updated,*
$$\left\|\Sigma_h-\mathbb{E}_{\pi\sim\mathit{Unif}(\Pi_h),s_h\sim\mathcal{D}_h^\pi,a_h\sim\rho_{s_h}}\left[\phi(s_h,a_h)\phi(s_h,a_h)^\top\right]\right\|_2\le\epsilon_2. \tag{10}$$
2. *Ω_2: Each time θ_h is updated,*
$$\mathbb{E}_{\pi\sim\mathit{Unif}(\Pi_h),s\sim\mathcal{D}_h^\pi,a\sim\rho_s}\left[\left((\theta_h-\theta_h^*)^\top\phi(s,a)\right)^2\right]\le 4\left(\eta+\epsilon_1+\lambda_{\mathrm{ridge}}\right), \tag{11}$$
where η is defined as in Lemma 7.
3. *Ω_3: Each time θ_h is updated,*
$$\mathbb{E}_{\pi\sim\mathit{Unif}(\Pi_h),s\sim\mathcal{D}_h^\pi,a\sim\rho_s}\left[\left((\theta_h-\theta_h^*)^\top\phi(s,a)\right)^2\right]\le 288\eta^{1.5}C^{2.5}d^{4.5}\left(\frac{2B}{\epsilon}\right)^{0.5}, \tag{12}$$
where η and C are defined as in Lemma 14.

Then under Assumption 3, $\Pr[\Omega_1\cap\Omega_2]\ge 1-\epsilon$. Alternatively, under Assumption 4, $\Pr[\Omega_1\cap\Omega_3]\ge 1-\epsilon$.

Proof. Note that $N\ge\frac{d\log(1/\epsilon_2)}{\epsilon_2^2}$ where $\epsilon_2\le\frac{\epsilon^6}{d}$. Therefore, by Lemma 6, each time Σ_h is updated, (10) holds with probability at least $1-\epsilon^2$.

As for (11), note that $N\ge\frac{d\log(1/\epsilon_2)}{\epsilon_2^2}\gg\frac{d}{\epsilon_1^2}\cdot\log(\frac{d}{\epsilon^2})$. Thus by Lemma 7, each time θ_h is updated, (11) holds with probability at least $1-\epsilon^2$.

Similarly, for (12), under the choice of parameters under Assumption 4, $N \geq \frac{d}{\epsilon_2^3} \gg \left(\frac{d}{\epsilon_2^2} + \frac{1}{\eta}\right) \ln \frac{2dB}{\epsilon} + \frac{2B}{\epsilon}$. Thus by Lemma 14, the probability that (12) is violated each step is at most $\epsilon/2B$.

Note that when the algorithm terminates, the Σ_h and θ_h are updated at most $|\Pi_h|$ times. Also note that, if during the first B updates, neither (10) nor (11) are violated, by Lemma 10 it follows that $|\Pi_h| \leq B$ when the algorithm terminates. In other words,

$$\Pr[\Omega_1 \cup \Omega_2] \geq 1 - B \cdot 2\epsilon^2 \geq 1 - \epsilon.$$

Similarly, under Assumption 4,

$$\Pr[\Omega_1 \cup \Omega_3] \geq 1 - B \cdot \epsilon^2 - B \cdot \frac{\epsilon}{2B} \geq 1 - \epsilon.$$

□

Lemma 9 (Distribution shift error checking)**.** *Assume that $\epsilon_2 < \min\{\frac{1}{2}\beta\lambda_r, \frac{\lambda_r}{2B}\}$. Consider the* `if` *clause when checking for $\pi_h \in \Pi_h$, i.e. when computing $\|\Sigma_{h'}^{-\frac{1}{2}} \hat{\Sigma}_{h'} \Sigma_{h'}^{-\frac{1}{2}}\|_2$. Define*

$$M_1 := \frac{\lambda_r}{|\Pi_{h'}|} I + \mathbb{E}_{\pi \sim \textit{Unif}(\Pi_{h'}), s_{h'} \sim \mathcal{D}_{h'}^{\pi}, a_{h'} \sim \rho_{s_{h'}}} \left[\phi(s_{h'}, a_{h'}) \phi(s_{h'}, a_{h'})^\top\right],$$

and

$$M_2 := \mathbb{E}_{s_{h'} \sim \mathcal{D}_{h'}^{\tilde{\pi}_h}, a_{h'} \sim \rho_{s_{h'}}} \left[\phi(s_{h'}, a_{h'}) \phi(s_{h'}, a_{h'})^\top\right].$$

Then under the event Ω defined in Lemma 8, when $\|\Sigma_{h'}^{-\frac{1}{2}} \hat{\Sigma}_{h'} \Sigma_{h'}^{-\frac{1}{2}}\|_2 \leq \beta|\Pi_{h'}|$,

$$\|M_1^{-1/2} M_2 M_1^{-1/2}\|_2 \leq 3\beta|\Pi_{h'}|.$$

When $\|\Sigma_{h'}^{-\frac{1}{2}} \hat{\Sigma}_{h'} \Sigma_{h'}^{-\frac{1}{2}}\|_2 \geq \beta|\Pi_{h'}|$,

$$\|M_1^{-1/2} M_2 M_1^{-1/2}\|_2 \geq \frac{1}{4}\beta|\Pi_{h'}|.$$

Proof. By Lemma 6,

$$\|M_1 - \Sigma_{h'}\|_2 \leq \epsilon_2 \leq \frac{\lambda_r}{2B} \leq \frac{1}{2}\lambda_{\min}\left(\Sigma_{h'}\right).$$

Thus $\frac{1}{2}\Sigma_{h'} \preccurlyeq M_1 \preccurlyeq 2\Sigma_{h'}$. Also by Lemma 6, $\|M_2 - \hat{\Sigma}_{h'}\|_2 \leq \epsilon_2$. Therefore, if $\|\Sigma_{h'}^{-\frac{1}{2}} \hat{\Sigma}_{h'} \Sigma_{h'}^{-\frac{1}{2}}\|_2 \geq \beta|\Pi_{h'}|$,

$$\begin{aligned}\|M_1^{-1/2} M_2 M_1^{-1/2}\|_2 &\geq \frac{1}{2}\|\Sigma_{h'}^{-1/2} M_2 \Sigma_{h'}^{-1/2}\|_2 \geq \frac{1}{2}\|\Sigma_{h'}^{-1/2} \hat{\Sigma}_{h'} \Sigma_{h'}^{-1/2}\|_2 - \frac{1}{2}\epsilon_2 \|\Sigma_{h'}^{-1}\|_2 \\ &\geq \frac{1}{2}\beta|\Pi_{h'}| - \frac{1}{2}\epsilon_2 \cdot \frac{|\Pi_{h'}|}{\lambda_r} \geq \frac{1}{4}\beta|\Pi_{h'}|.\end{aligned}$$

Similarly, when $\|\Sigma_{h'}^{-\frac{1}{2}} \hat{\Sigma}_{h'} \Sigma_{h'}^{-\frac{1}{2}}\|_2 \leq \beta|\Pi_{h'}|$,

$$\begin{aligned}\|M_1^{-1/2} M_2 M_1^{-1/2}\|_2 &\leq 2\|\Sigma_{h'}^{-1/2} M_2 \Sigma_{h'}^{-1/2}\|_2 \leq 2\|\Sigma_{h'}^{-1/2} \hat{\Sigma}_{h'} \Sigma_{h'}^{-1/2}\|_2 + 2\epsilon_2 \|\Sigma_{h'}^{-1}\|_2 \\ &\leq 2\beta|\Pi_{h'}| + 2\epsilon_2 \cdot \frac{|\Pi_{h'}|}{\lambda_r} \leq 3\beta|\Pi_{h'}|.\end{aligned}$$

□

Lemma 10 (Lemma A.6 in Du et al. [2019])**.** *Under the event Ω_1 defined in Lemma 8, $|\Pi_h| \leq B$ for all $h \in [H]$.*

Proof. We provide a proof for completeness. Fix a level $h' \in [H]$. Define

$$A := \lambda_r I + \sum_{\pi \in \Pi_{h'}} \mathbb{E}_{s_{h'} \sim \mathcal{D}_{h'}^{\tilde{\pi}_h}, a_{h'} \sim \rho_{s_{h'}}} [\phi(s_{h'}, a_{h'}) \phi(s_{h'}, a_{h'})^\top].$$

By the update rule at Line 6, $|\Pi_{h'}|$ is expanded if and only if the `if` clause at Line 5 returns False when checking for some $\tilde{\pi}_h$. By Lemma 9, define

$$M := \mathbb{E}_{s_{h'}\sim \mathcal{D}_{h'}^{\tilde{\pi}_h}, a_{h'}\sim \rho_{s_{h'}}} \left[\phi(s_{h'}, a_{h'})\phi(s_{h'}, a_{h'})^\top\right],$$

then

$$\|A^{-1/2} M A^{-1/2}\|_2 \geq \frac{1}{4}\beta = 2.$$

Note that after $\Pi_{h'}$ is updated to $\Pi_{h'} \cup \{\tilde{\pi}_h\}$, A would be updated to $A + M$. Observe that

$$\det(A+M) = \det(A)\cdot \det\left(I + A^{-1/2} M A^{-1/2}\right) \geq 3\det(A).$$

Therefore during the execution of the algorithm,

$$\det(A) \geq 3^{|\Pi_{h'}|}\cdot \lambda_r^d.$$

On the other hand, since $\|\phi(s,a)\phi(s,a)^\top\|_2 \leq 1$,

$$\det(A) \leq (\lambda_r + |\Pi_{h'}|)^d.$$

The lemma follows by solving $3^{|\Pi_{h'}|}\cdot \lambda_r^d \leq (\lambda_r + |\Pi_{h'}|)^d$. □

6 Analysis of Ridge Regression under Hypercontractivity

Recall that a distribution $\mathcal{D}$ is $(C,4)$-hypercontractive if $\forall v$,

$$\mathbb{E}_{x\sim\mathcal{D}}[(x^\top v)^4] \leq C\cdot \left(\mathbb{E}_{x\sim\mathcal{D}}[(x^\top v)^2]\right)^2.$$

In this section we prove an strengthened version of Lemma 7 for hypercontractive distributions (Lemma 14), which may be of independent interest.

Lemma 11. *Let x be a d-dimensional r.v. If the distribution of x is $(C,4)$-hypercontractive and isotropic (i.e. $\mathbb{E}[xx^\top] = I$), then*

$$\Pr[\|x\|_2 > t] \leq \frac{Cd^2}{t^4}.$$

Proof. Consider a Gaussian random vector $v \sim N(0,I)$. Then

$$\mathbb{E}_v[(x^\top v)^4] = \|x\|^4 \cdot \mathbb{E}_{\xi\sim N(0,1)}\xi^4 = 3\|x\|^4.$$

Therefore

$$\begin{aligned}\mathbb{E}_x[\|x\|^4] &= \frac{1}{3}\mathbb{E}_{x,v}[(x^\top v)^4] \leq \frac{C}{3}\mathbb{E}_v\left(\mathbb{E}_x(x^\top v)^2\right)^2 \\ &\leq \frac{C}{3}\mathbb{E}_v\|v\|^4 = \frac{C\cdot(d^2+2d)}{3} \leq d^2 C.\end{aligned}$$

The claim then follows from Markov's inequality. □

Lemma 12. *If the $x_1,\cdots,x_n$ are i.i.d. samples from a $(C,4)$-hypercontractive distribution. Let $\sigma(\cdot)$ denote the decreasing order of $\|x_i\|_2$. Then with probability $1-\delta$,*

$$\sum_{k=1}^m \|x_{\sigma(k)}\|_2 = 3\delta^{-1/4} n^{1/4} m^{3/4} C^{1/4} d^{1/2}.$$

Proof. Fix $k\in[m]$. Set $t = \alpha\left(\frac{Cd^2 n}{k}\right)^{1/4}$. By Lemma 11,

$$\begin{aligned}\Pr[\|x_{\sigma(k)}\|_2 > t] &\leq \binom{n}{k}\Pr[\|x\| > t]^k \leq \binom{n}{k}\cdot\left(\frac{Cd^2}{t^4}\right)^k \\ &\leq \frac{n^k}{k!}\cdot\frac{k^k}{\alpha^{4k} n^k} \leq \left(\frac{e}{\alpha^4}\right)^k.\end{aligned}$$

Choosing $\alpha = \left(\frac{2e}{\delta}\right)^{1/4}$ gives $\Pr[\|x_{\sigma(k)}\|_2 > t] \le (\delta/2)^k$. By a union bound, with probability $1-\delta$,

$$\sum_{i=1}^{m} \|x_{\sigma(i)}\|_2 \le \sum_{k=1}^{m} (2e/\delta)^{1/4} \left(\frac{Cd^2 n}{k}\right)^{1/4} \le 3\delta^{-1/4} n^{1/4} m^{3/4} C^{1/4} d^{1/2}.$$

□

Lemma 13 (Lemma 3.4 Bakshi and Prasad [2020]). *If $\mathcal{D}$ is $(C,4)$-hypercontractive and $x_1, \cdots, x_n$ are i.i.d. samples drawn from $\mathcal{D}$. Let $\Sigma := \mathbb{E}_{x\sim\mathcal{D}}[xx^\top]$. With probability $1-\delta$,*

$$\left(1 - \frac{Cd^2}{\sqrt{n\delta}}\right)\Sigma \preccurlyeq \frac{1}{n}\sum_{i=1}^{n} x_i x_i^\top \preccurlyeq \left(1 + \frac{Cd^2}{\sqrt{n\delta}}\right)\Sigma.$$

Lemma 14 (Risk bound for ridge regression with hypercontractivity). *Suppose that $(x_1, y_1), \cdots, (x_N, y_N)$ are i.i.d. data drawn from $\mathcal{D}$ with*

$$y_i = \theta^\top x_i + b_i + \xi_i,$$

where $\Pr[b_i \neq 0] \le \eta$, $\|b\|_\infty \le 1$, $|\xi_i| \le 1$, and $\mathbb{E}[\xi_i] = 0$. Assume that distribution of x is $(C,4)$-hypercontractive (see Assumption 4). Let the ridge regression estimator be

$$\hat{\theta} = \left(\sum_{i=1}^{N} x_i x_i^\top + N\lambda_{\text{ridge}} \cdot I\right)^{-1} \cdot \sum_{i=1}^{N} x_i y_i.$$

If $N = \Omega\left((\frac{d}{\epsilon_N^2} + \frac{1}{\eta})\log(\frac{d}{\delta}) + \frac{1}{\delta}\right)$, then with probability at least $1-\delta$,

$$\mathbb{E}_{x\sim\mathcal{D}}\left[\left((\hat{\theta}-\theta)^\top x\right)^2\right] \le 8\left(\epsilon_N + \lambda_{\text{ridge}}\right) + 288\eta^{1.5} C^{2.5} d^{4.5} \delta^{-0.5}.$$

Proof. Define $\hat{\Sigma} := \frac{1}{N}\sum_{i=1}^{N} x_i x_i^\top$ and $\Sigma := \mathbb{E}_{x\sim\mathcal{D}}[xx^\top]$. Then

$$\begin{aligned}
\hat{\theta} &= \frac{1}{N}\left(\lambda_{\text{ridge}} I + \hat{\Sigma}\right)^{-1} \sum_{i=1}^{N} \left(x_i x_i^\top \theta + x_i \cdot \xi_i + x_i \cdot b_i\right) \\
&= \underbrace{\frac{1}{N}\left(\lambda_{\text{ridge}} I + \hat{\Sigma}\right)^{-1} \sum_{i=1}^{N} b_i x_i}_{(a)} + \underbrace{\frac{1}{N}\left(\lambda_{\text{ridge}} I + \hat{\Sigma}\right)^{-1} \sum_{i=1}^{N} \left(x_i x_i^\top \theta + x_i \cdot \xi_i\right)}_{(b)}.
\end{aligned}$$

By Lemma 7, $\|\theta - (b)\|_\Sigma^2 \le 4(\epsilon_N + \lambda_{\text{ridge}})$. It remains to bound the $\|\cdot\|_\Sigma$ norm of (a).

First, by Hoeffding's inequality, with probability $1-\delta$, $\|b\|_0 = \sum_{i=1}^{n} I[b_i \neq 0] \le 2\eta N$. Define $z_i := \Sigma^{-1/2} x_i$ to be the normalized input. It can be seen that $\mathbb{E}[z_i z_i^\top] = I$ and that the distribution of z_i is also hypercontractive. By Lemma 12, with probability $1-2\delta$,

$$\sum_{i=1}^{n} \|z_i\|_2 \cdot I[b_i \neq 0] \le 3\delta^{-1/4} N^{1/4} (2\eta N)^{3/4} (Cd^2)^{1/4}.$$

It follows that with probability $1-2\delta$,

$$\begin{aligned}
\|(a)\|_\Sigma = \frac{1}{N}\left\|\hat{\Sigma}^{-1}\sum_{i=1}^{N} x_i b_i\right\|_\Sigma &\le \frac{1}{N}\sum_{i=1}^{N} \|\Sigma^{1/2}\hat{\Sigma}^{-1} x_i b_i\|_2 \\
&= \frac{1}{N}\sum_{i=1}^{N} \|\Sigma^{1/2}\hat{\Sigma}^{-1}\Sigma^{1/2} z_i b_i\|_2 \\
&\le \frac{1}{N}\|\Sigma^{1/2}\hat{\Sigma}^{-1}\Sigma^{1/2}\|_2 \cdot \sum_{i=1}^{N} \|z_i\|_2 \cdot H \cdot I[b_i \neq 0] \\
&\le 3H\left(1 + \frac{Cd^2}{\sqrt{N\delta}}\right) \cdot \delta^{-1/4} N^{-3/4} (2\eta N)^{3/4} (Cd^2)^{1/4} \\
&\le 12H\eta^{0.75} \cdot C^{\frac{5}{4}} d^{\frac{9}{4}} \delta^{-\frac{1}{4}}.
\end{aligned}$$

Therefore

$$\begin{aligned}\|\hat{\theta} - \theta\|_\Sigma^2 &\leq 2\|\hat{\theta} - (b)\|_\Sigma^2 + 2\|(a)\|_\Sigma^2 \\ &\leq 8(\epsilon_N + \lambda_{\text{ridge}}) + 288\eta^{1.5}C^{2.5}d^{4.5}\delta^{-0.5}.\end{aligned}$$

□

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