
Faster Directional Convergence of Linear Neural Networks under Spherically Symmetric Data

Dachao Lin¹ Ruoyu Sun² Zhihua Zhang³

¹Academy for Advanced Interdisciplinary Studies, Peking University

²Department of Industrial and Enterprise Engineering, Coordinate Science Lab (affiliated)
University of Illinois Urbana-Champaign

³School of Mathematical Sciences, Peking University

lindachao@pku.edu.cn ruoyus@illinois.edu zhzhang@math.pku.edu.cn

Abstract

In this paper, we study gradient methods for training deep linear neural networks with binary cross-entropy loss. In particular, we show global directional convergence guarantees from a polynomial rate to a linear rate for (deep) linear networks with spherically symmetric data distribution, which can be viewed as a specific zero-margin dataset. Our results do not require the assumptions in other works such as small initial loss, presumed convergence of weight direction, or overparameterization. We also characterize our findings in experiments.

1 Introduction

Local v.s. non-local analysis. Deep neural networks have been successfully trained with simple gradient-based methods, despite the inherent non-convexity of the objective function. Recently, a number of works proved the convergence of gradient methods to global minima for ultra-wide neural networks [1, 4, 12, 13, 20, 31]. These works essentially performed “local analysis” because in their proofs the parameters stay close to initialization during training. In the large width setting, there exists a global minimum near a random initial point, thus staying in the local region can still result in convergence to global minima. Despite the technical convenience of handling a local region, in practical training, the parameters often travel far from initialization (see, e.g., [15]). To understand the practical optimization trajectory, it is important to develop a *non-local* convergence analysis.

Deep linear nets with quadratic loss: “local analysis”. To understand the behavior of gradient descent (GD) for general neural nets, we need to understand GD for deep linear nets. A number of works analyzed linear networks with the square loss. Bartlett et al. [7] provided that gradient descent converges to the target matrix at a linear rate from identity initialization, while assuming the target matrix is either close to identity or positive definite. Arora et al. [2] proved linear convergence of deep linear networks if the initialization has a positive “deficiency margin” and is nearly balanced. Later, a few works followed a similar idea to neural tangent kernel (NTK) [20] to establish convergence analysis. Du and Hu [11] showed GD with Gaussian random initialization converges to a global minimum at a linear rate if the width of every hidden layer is large enough. Hu et al. [19] improved the width requirement to be independent of depth, by utilizing orthogonal weight initialization, but they assume each layer to have the same width. All the above works are not non-local analysis.

Deep linear nets with quadratic loss: non-local analysis. Eftekhari [14] provided non-local convergence analysis for deep linear nets with quadratic loss. When one layer has only one neuron (including scalar output case) and assuming that the input data are whitened, they proved that gradient flow converges to global minimizers starting from balanced initialization.

In this work, we are interested in the classification task with binary cross-entropy loss. To our knowledge, there was no non-local convergence analysis of gradient flow for deep linear nets in such a scheme.

Deep linear nets with logit loss: final-phase analysis. Several recent works [17, 38, 23, 21, 37] studied the exponential-type loss for deep linear networks. Specifically, Gunasekar et al. [17], Nacson et al. [38] proved the convergence to max-margin solution, but under the assumption that the weight direction and the loss have converged to global optima, which is a “final-point analysis”. Lyu and Li [37], Ji and Telgarsky [21, 23] also proved the convergence to max-margin solution under the assumption that the initial point has already obtained zero classification error (weaker assumption than [17, 38]), i.e., they analyzed the “final phase” of training. In this work, we would like to understand the entire training dynamics, not just the “late training” period.

1.1 Our Contributions

In this paper, we analyze gradient flow for deep linear networks with logit loss (i.e. binary cross-entropy loss). The main contributions of this paper are summarized as follows:

- **Convergence result.** We prove the global convergence of gradient flow for minimizing a population logit loss with deep linear nets, under the assumption that the input data is spherically symmetric (Theorem 3). This assumption covers the standard Gaussian distribution and uniform distribution on the sphere. To our knowledge, this is the first global analysis of gradient flow on deep linear nets with logit loss, though under certain strong assumptions. We emphasize that our analysis is beyond the “lazy training” scheme.
- **Convergence rate.** We also establish explicit convergence rate of gradient flow (Theorem 3). Denote $\theta(t)$ as the angle between $\mathbf{w}_e(t)$ and \mathbf{v} , where $\mathbf{w}_e(t)$ is the collection of parameters at time t following gradient flow, and \mathbf{v} is the directional global minimizer. In the first phase, we have polynomial convergence rate of $\cos(\theta(t)) - 1 = O((N_1 t)^{-\frac{2}{N_1+2}})$, where $N_1 + 2$ is the depth of the deep linear network. In the second phase, we have linear convergence rate of $\cos(\theta(t)) - 1 = e^{-O(t)}$. And the second phase begins when the induced weight norm changes from descending to ascending mentioned below.
- **Weight norm change pattern.** We prove that the induced weight norm goes through descending and ascending periods. If the initial induced weight norm starts with descending behavior, then after finite time, it will change to ascending and continues increasing to infinity. If the initial induced weight norm begins with ascending behavior, then it would increase to infinity directly.

We also verify our results in numerical experiments including the descending and ascending behavior of weight norm and the convergence rates in our setting.

1.2 Additional Related work

Directional convergence for exponentially-tailed loss. Previously, we mentioned a few works on deep linear nets for classification. Next, we discuss convergence results on classification for more general settings (not just deep linear nets). For binary classification problems, oftentimes it is the direction of the predictor that matters, while the norm is less important. For a finite number of linearly separable data, there can be infinitely many directions that lead to zero loss, and classical optimization results do not show which one GD converges to. In many recent works [22, 24, 25, 38, 39, 46, 16], it is shown that for binary classification problems with exponential-type loss (which can be viewed as the 1-layer 1-linear-neuron net case) and linearly separable data, GD converges to the ℓ_2 maximum margin direction. These works assume there is a positive margin. Ji and Telgarsky [22], Ji et al. [25] also consider a non-separable case, but still requires a positive margin in the “separable part”. Moreover, some works [21, 23, 17, 38] study the convergence of GD for training deep linear nets. Lyu and Li [37] further studied deep homogeneous networks (covering deep linear nets and deep ReLU nets). Yun et al. [51] provided a unified framework that connects several existing results under a general tensor formulation of linear networks. A few other works study wide non-linear neural nets, e.g., Chizat and Bach [9], Bach and Chizat [5] improved previous results in the infinitely wide two-layer case, identifying the learned classifier as the solution of a convex max-margin problem.

There are two differences between those works and ours. First, none of these works have performed non-local convergence analysis for deep linear nets as the current work does. Second, they consider finite data with a positive margin, and we consider infinitely many data with a zero margin. We underline that previous convergence rates are inversely proportional to such a margin, which is vacuous in our zero-margin setting.

More discussions on non-local analysis of deep linear nets with quadratic loss. We mentioned Eftekhari [14] earlier; here we add more discussions. Besides the main difference in the loss functions, there are two other differences between Eftekhari [14] and our work. First, they assume whitened data, while we assume spherically symmetric data. Second, their convergence rate result requires an extra assumption that the inverse spectral gap of the target matrix is small ([14, Theorem 4.4]), while our convergence rate result does not require extra assumption. Besides the work [14], another earlier work [6] performed “non-local analysis”, but as pointed out by [14], their work only ensures convergence to the minimizer in a constrained subset, not necessarily the global minimizer (see Appendix F in [14] for detail).

Global landscape analysis. In our work, the term “non-local analysis” should actually be called “non-local *convergence* analysis”. There is another line of works on “non-local *landscape* analysis” (see, e.g., [26, 36, 27, 42, 52, 41, 30, 10, 34, 33, 35, 49, 44] and the surveys [48, 47]) which analyze the properties of stationary points, local minima, strict local minima, etc. These works do not directly analyze the convergence of GD, thus are different from our work.

2 Preliminaries

We consider a binary classification problem where the data are generated as follows. Let $\mathbf{v} \in \mathbb{R}^d$ be a fixed vector satisfying $\|\mathbf{v}\| = 1$. The input data point $\mathbf{x} \in \mathbb{R}^d$ is sampled from a distribution D and the label for each \mathbf{x} is $y(\mathbf{x}) = \text{sgn}(\mathbf{v}^\top \mathbf{x}) \in \{-1, +1\}$.

Suppose we want to learn the parameter \mathbf{w} for a predictor $\phi(\cdot, \mathbf{w}) : \mathbb{R}^d \rightarrow \mathbb{R}$ (the specific form of ϕ will be discussed later). We use the population logit loss $\ell(z) := \ln(1 + e^{-z})$ (binary cross-entropy) with $z(\mathbf{x}, \mathbf{w}) = y(\mathbf{x}) - \phi(\mathbf{x}, \mathbf{w})$. The parameter is learned by solving the optimization problem

$$\min_{\mathbf{w}} L(\mathbf{w}) := \mathbb{E}_{\mathbf{x} \sim D} \ln \left(1 + e^{-y(\mathbf{x})\phi(\mathbf{x}, \mathbf{w})} \right). \quad (1)$$

We focus on the following standard gradient methods.

1) **Gradient flow:** We initialize $\mathbf{w}(0)$, and for every $t > 0$ let $\mathbf{w}(t)$ be the solution of the differential equation: $\dot{\mathbf{w}}(t) = -\nabla L(\mathbf{w}(t))$.

2) **Gradient descent:** We initialize $\mathbf{w}(0)$ and set a sequence of positive learning rates $\{\eta_k\}_{k=1}^\infty$. At each iteration $n > 0$, we do a single step in the negative direction of the gradient, that is, $\mathbf{w}(k+1) = \mathbf{w}(k) - \eta_k \nabla L(\mathbf{w}(k))$.

Notation. We denote vectors by lowercase bold letters (e.g., $\mathbf{u}, \mathbf{x}, \mathbf{v}$). Moreover, $\|\cdot\|_k$ denotes the standard ℓ_2 -norm for a vector, or the spectral norm for a given matrix. Let $\Sigma = \mathbb{E}_{\mathbf{x} \sim D} \mathbf{x}\mathbf{x}^\top$ be the population covariance matrix, and $\lambda_{\max}(A)$ be the maximum eigenvalue of a real symmetric matrix A . We by S^{d-1} denote the surface of an d -dimensional unit sphere, and by \mathbf{I}_k the k -dimensional identity matrix.

We let $\bar{\mathbf{w}} := \mathbf{w}/\|\mathbf{w}\|_k$ whenever $\|\mathbf{w}\|_k \neq 0$. Given vectors $\mathbf{w} = (w_1, \dots, w_d)^\top \in \mathbb{R}^d$ and $\mathbf{v} = (v_1, \dots, v_d)^\top \in \mathbb{R}^d$, we let $\theta(\mathbf{w}, \mathbf{v}) := \arccos[\mathbf{w}^\top \mathbf{v} / (\|\mathbf{w}\|_k \|\mathbf{v}\|_k)] \in [0, \pi]$ denote the angle between \mathbf{w} and \mathbf{v} , $\theta(t) := \theta(\mathbf{w}(t), \mathbf{v})$ and $\theta(n) := \theta(\mathbf{w}(n), \mathbf{v})$. We let $D_{\mathbf{w}, \mathbf{v}}$ be the marginal distribution of $\mathbf{x} = (x_1, \dots, x_d)^\top$ on the subspace spanned by two linearly independent vectors \mathbf{w}, \mathbf{v} (as a distribution over \mathbb{R}^2), and $D_{\mathbf{v}}$ similarly. Let $D_2 := D_{\mathbf{e}_1, \mathbf{e}_2}$ where $\mathbf{e}_1, \dots, \mathbf{e}_d$ are the d -dimensional coordinate directions, and $c_0 := \mathbb{E}_{\mathbf{x} \sim D_2} \|\mathbf{x}\|_k$.

Additionally, we use the standard $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ to hide universal constant factors. We say the algorithm achieves *superpolynomial* convergence if it takes $O(\ln^\alpha(1/\epsilon))$ iterations (for a constant $\alpha > 0$) to achieve ϵ -error.

¹Here, $\text{sgn}(z) = 1$ if $z \geq 0$, otherwise -1 .

2.1 Data distribution assumption

Assumption 1 D is a spherically symmetric distribution, i.e., if $\mathbf{x} \sim D$, then for any orthogonal matrix A , we have $A\mathbf{x} \sim D$.

Unlike the common finite dataset, our main assumption on the data is the above. Such construction gives a zero-margin separable dataset in our scheme. Moreover, our assumption includes the standard Gaussian distribution (adopted in [8, 53, 45, 54, 32]), and uniform distribution on a sphere S^{d-1} . Under this assumption, $D_{\mathbf{w}, \mathbf{v}}$ shares the same distribution for different pairs of $\mathbf{w}, \mathbf{v} \in \mathbb{R}^d$ as long as \mathbf{w} and \mathbf{v} are not parallel (i.e., $\mathbf{w} \perp \mathbf{v}$).

3 Warm-up: Logistic Regression Case

In this section, we analyze the logistic regression setting, i.e., $\phi(\mathbf{x}, \mathbf{w}) = \mathbf{w}^\top \mathbf{x}$. All proofs of this section are given in Appendix A. For the linear classifier, we have the gradient expression as follows

$$\nabla L(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim D} \frac{y(\mathbf{x})\mathbf{x}}{1 + e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}}. \quad (2)$$

3.1 Convergence of gradient flow for logistic regression

Our main result about logistic regression is the following. The proof is given in Appendix A.6.

Theorem 1 Consider gradient flow for solving the problem (1). Denote $T = \inf \{t : \frac{\partial k\mathbf{w}(t)k^2}{\partial t} \leq 0\}$. Under Assumption 1, we have

$$\cos \theta(t) \begin{cases} 1 - \frac{2}{e^{A_1 t + B_1} + 1}, & t \leq T, \\ 1 - \frac{2}{e^{A_2 (t - T) + C_2 + B_2} + 1}, & t > T. \end{cases}$$

Here $A_1 = \frac{2c_0}{\pi k\mathbf{w}(0)k} > 0$, $B_1 = 2 \ln \left| \tan \frac{\theta(0)}{2} \right|$, $A_2 = \frac{4c_0}{0.6\pi} > 0$, $B_2 = 2 \ln \left| \tan \frac{\theta(T)}{2} \right| - \frac{4c_0 k\mathbf{w}(T)k}{0.6\pi}$, and $C_2 = \frac{k\mathbf{w}(T)k^2}{0.6}$.

This result shows that as $t \rightarrow \infty$, the error metric $1 - \cos \theta(t)$ converges to 0, which implies $\mathbf{w}(t) \rightarrow \mathbf{v}$, i.e., the weight converges to the globally optimal direction. Additionally, the training goes through two stages: in the first stage, $1 - \cos \theta(t) = O(e^{-A_1 t})$; in the second stage, $1 - \cos \theta(t) = O(e^{-A_2 (t - T)})$.

Next, we present an interesting characterization of the optimization trajectory: the weight norm goes through a descending stage and an ascending stage. This “descending-ascending pattern” can be established for deep linear nets as well (see Theorem 3 later).

Lemma 1 Denote $T := \inf \{t : \frac{\partial k\mathbf{w}(t)k^2}{\partial t} \leq 0\}$. Under Assumption 1, we have $T < \infty$, and

$$\frac{\partial k\mathbf{w}(t)k^2}{\partial t} \begin{cases} < 0, & t \geq T. \\ > 0, & t < T. \end{cases} \quad (3)$$

This lemma shows that the training dynamics can be divided into two stages: in the first stage, the norm $k\mathbf{w}(t)k$ is decreasing; in the second stage, the norm $k\mathbf{w}(t)k$ is increasing. The first stage might be non-existent if $\frac{\partial k\mathbf{w}(0)k^2}{\partial t} \leq 0$, and the second stage is always existent (i.e., $T < \infty$ as stated in the lemma). The existence of the second stage implies that there is a lower bound of the weight norm throughout training.

Remark 1 The lower bound of weight norm is not needed for establishing Theorem 1 (the convergence result in the logistic regression case), but will be crucial for multi-layer linear nets. The reason we present Lemma 1 is that its analysis is essentially similar to the multi-layer nets case while easier to understand.

3.2 Basic properties for the proof of Theorem 1

In this subsection, we describe a few technical propositions that will be used in the proof of Theorem 1. Readers who care about the proof sketch can skip this part, and turn back to check these propositions if we mention subsequently.

Proposition 1 Denote $c := \mathbb{E}_{\mathbf{x} \sim D} \|\mathbf{x}\|$. Suppose $\phi(\mathbf{x}, \mathbf{w}) = \mathbf{w}^\top \mathbf{x}$ and $P(\mathbf{f}_{\mathbf{x}} : \mathbf{v}^\top \mathbf{x} = 0) = 0$. Then the function $L(\mathbf{w})$ defined in Eq. (1) is c -Lipschitz continuous, $\frac{1}{4} \lambda_{\max}(\cdot)$ -smooth, and convex but not strongly convex.

Generally speaking, gradient methods do not exhibit linear convergence rate for non-strongly convex functions [40]. However, the training dynamic enjoy faster convergence rates in terms of directional convergence, due to certain invariant properties (such as Proposition 2 below).

Proposition 2 Suppose $\phi(\mathbf{x}, \mathbf{w}) = \mathbf{w}^\top \mathbf{x}$ and $P(\mathbf{f}_{\mathbf{x}} : \mathbf{v}^\top \mathbf{x} = 0) < 1$.

Then the gradient flow for the problem in Eq. (1) satisfies the following properties: (i) $\mathbf{v}^\top \nabla L(\mathbf{w}) < 0$ for any $\mathbf{w} \in \mathbb{R}^d$. (ii) $\mathbf{v}^\top \mathbf{w}(t)$ is increasing as t increases; (iii) $\|\nabla L(\mathbf{w}(t))\|$ is unbounded;

Similarly, the gradient descent for the problem in Eq. (1) satisfies the following properties: (iv) if the learning rates are lower bounded, i.e., $\eta_n \geq \eta > 0$, then $\mathbf{v}^\top \mathbf{w}(n)$ is increasing as n increases, and $\|\nabla L(\mathbf{w}(n))\|$ is unbounded.

When Assumption 1 holds, we can provide a more precise description of $\mathbf{v}^\top \mathbf{w}(t)$ and $\nabla L(\mathbf{w})$ below:

Proposition 3 Under Assumption 1, we have: (i) $\|\nabla L(\mathbf{w})\| \leq c_0 = \mathbb{E}_{\mathbf{x} \sim D} \|\mathbf{x}\|$; (ii) $\nabla L(\alpha \mathbf{v}) = \mathbb{E}_{t \sim D} \frac{jt}{1+e^{\alpha jt}} \mathbf{v}$; (iii) $\lim_{t \rightarrow \infty} \mathbf{v}^\top \mathbf{w}(t) = +1$. (iv) If the learning rates are lower bounded, i.e., $\eta_n \geq \eta > 0$, then $\lim_{n \rightarrow \infty} \mathbf{v}^\top \mathbf{w}(n) = +1$.

Remark 2 This proposition shows that in the logistic regression case, we do not need to consider the case $\mathbf{w}(t) \perp \mathbf{v}$ (including $\mathbf{0}$), since we have $\nabla L(\mathbf{w}(t)) \perp \mathbf{v}$ when $\mathbf{w}(t) \perp \mathbf{v}$ by (ii) in Proposition 3. Therefore, we only need to consider the case $\mathbf{w}(t) \neq \mathbf{0}$ and $\mathbf{w}(t) \cdot \mathbf{v} > 0$.

3.3 Proof sketch of Theorem 1

We mainly consider the dynamics of $\cos \theta(\mathbf{w}, \mathbf{v})$, which is also employed in [29, 28, 50, 43].

$$\frac{\partial \cos \theta(t)}{\partial t} = \underbrace{\frac{1}{\|\nabla L(\mathbf{w}(t))\|}}_{I_1} \underbrace{(\mathbf{v} - \frac{(\mathbf{v}^\top \nabla L(\mathbf{w}(t)))}{\|\nabla L(\mathbf{w}(t))\|^2} \nabla L(\mathbf{w}(t)))^\top \nabla L(\mathbf{w}(t))}_{I_2}. \quad (4)$$

Step 1: We have the analytic expression of I_2 shown in Lemma 2 below. This expression relies on Assumption 1, because the symmetry eliminates the collision among data points.

Lemma 2 Under Assumption 1 and if $\mathbf{w} \neq \mathbf{0}$, then

$$(\mathbf{v} - \frac{(\mathbf{v}^\top \nabla L(\mathbf{w}))}{\|\nabla L(\mathbf{w})\|^2} \nabla L(\mathbf{w}))^\top \nabla L(\mathbf{w}) = \mathbf{v}^\top (\mathbf{I}_d - \frac{\nabla L(\mathbf{w}) \nabla L(\mathbf{w})^\top}{\|\nabla L(\mathbf{w})\|^2}) \nabla L(\mathbf{w}) = \frac{c_0 \sin^2 \theta(\mathbf{w}, \mathbf{v})}{\pi}. \quad (5)$$

Moreover, if additionally $\mathbf{w} \cdot \mathbf{v} > 0$, the above result implies

$$(\mathbf{I}_d - \frac{\nabla L(\mathbf{w}) \nabla L(\mathbf{w})^\top}{\|\nabla L(\mathbf{w})\|^2}) \nabla L(\mathbf{w}) = \frac{c_0}{\pi} (\mathbf{I}_d - \frac{\nabla L(\mathbf{w}) \nabla L(\mathbf{w})^\top}{\|\nabla L(\mathbf{w})\|^2}) \mathbf{v}. \quad (6)$$

From Lemma 2 and Eq. (4), we obtain that $\frac{\partial \cos \theta(t)}{\partial t} > 0$ when $\mathbf{w}(t) \neq \mathbf{0}$ and $\mathbf{w}(t) \cdot \mathbf{v} > 0$.

Step 2: Next, we analyze the remaining term $\|\nabla L(\mathbf{w}(t))\|$ in Eq. (4) by its derivative. Denote

$$g(\mathbf{w}(t)) := \frac{1}{2} \frac{\partial \|\nabla L(\mathbf{w}(t))\|^2}{\partial t} = \mathbf{w}(t)^\top \nabla L(\mathbf{w}(t)). \quad (7)$$

The following lemma describes the behavior of $g(\mathbf{w}(t))$. Some properties in Lemma 3 can be explained visually from Figure 2 in the Appendix.

Lemma 3 Under Assumption 1, the following hold:

(i) $g(\mathbf{w}) \leq 0.3$; (ii) If $\theta(\mathbf{w}, \mathbf{v}) > \pi/2$, then $g(\mathbf{w}) < 0$;

(iii) If $\theta(\mathbf{w}, \mathbf{v}) \leq \pi/2$, then $\partial g(\mathbf{w})/\partial \theta(\mathbf{w}, \mathbf{v}) < 0$.

(iv) If $0 < k\mathbf{w}k \leq \frac{c_0 \cos \theta(\mathbf{w}, \mathbf{v})}{2\pi L}$ with $L := \frac{1}{8} \mathbb{E}_{\mathbf{x} \sim \mathcal{D}_2} k\mathbf{x}k^2$, then $g(\mathbf{w}) \cos \theta(\mathbf{w}, \mathbf{v}) > 0$.

Step 3: Finally, we bound $k\mathbf{w}(t)k \leq k\mathbf{w}(0)k$ if $t \leq T$, and $k\mathbf{w}(t)k^2 \leq k\mathbf{w}(T)k^2 \leq 0.6(t - T)$ if $t > T$ from (i) in Lemma 3, which shows two-phase directional convergence in Theorem 1.

3.4 Convergence of gradient descent for linear regression

Now we turn to the gradient descent setting based on the previous results. The difficulty we encounter is that the arbitrary choice of learning rates in each step may break the directional monotonicity in Lemma 2 as Eq. (8) reveals (See Appendix A for the proof).

$$\cos \theta(n+1) - \cos \theta(n) = \frac{1}{k\mathbf{w}(n+1)k} \left[\eta_n (\mathbf{v} - \langle \bar{\mathbf{w}}(n), \mathbf{v} \rangle \bar{\mathbf{w}}(n))^{\top} \mathbf{r} L(\mathbf{w}(n)) - (k\mathbf{w}(n+1)k - \langle \bar{\mathbf{w}}(n), \mathbf{w}(n+1) \rangle) \cos \theta(n) \right]. \quad (8)$$

Fortunately, when $\mathbf{v}^{\top} \mathbf{w}(n) \leq 0$, we have $g(\mathbf{w}) \leq 0$ by (ii) in Lemma 3, implying that the first-phase directional convergence in the gradient flow case still holds.

Lemma 4 Under Assumption 1, suppose $\theta(0) \notin \pi$ and $\mathbf{v}^{\top} \mathbf{w}(0) < 0$. Then we have that

$$\cos \theta(n) \leq 1 - (1 - \cos \theta(0)) e^{-BS_n} \quad (9)$$

until $\cos \theta(n) \leq 0$, where $S_n := \sum_{k=0}^{n-1} \frac{\eta_k}{A + \sum_{i=0}^k \eta_i^2}$, $A = \frac{k\mathbf{w}(0)k^2}{c_0^2}$ and $B = \frac{1 + \cos \theta(0)}{\pi}$.

Hence, when $\mathbf{w}(n)$ stays in the ‘‘wrong’’ region that $\theta(n) > \pi/2$, larger learning rate gives faster directional convergence to the region $\bar{\mathbf{w}} : \theta(\bar{\mathbf{w}}, \mathbf{v}) \leq \pi/2$. Unfortunately, when $\theta(n) \leq \pi/2$, the directional dynamic becomes unstable and heavily relies on the current learning rate. However, after simple calculation invoked from Lemma 2, we find that (Lemma 12 in Appendix)

$$k\mathbf{w}(n+1)k^2 = (\langle \bar{\mathbf{w}}(n), \mathbf{w}(n+1) \rangle)^2 + \left(\frac{c_0 \eta_n}{\pi} \right)^2 \sin^2 \theta(n). \quad (10)$$

Combining Eq. (10) and Eq. (8), we figure out

$$\cos \theta(n+1) - \cos \theta(n) = \frac{1}{k\mathbf{w}(n+1)k} \left(\frac{c_0 \eta_n \sin^2 \theta(n)}{\pi} - \frac{c_0^2 \eta_n^2 \sin^2 \theta(n) \cos \theta(n) / \pi^2}{k\mathbf{w}(n+1)k + \langle \bar{\mathbf{w}}(n), \mathbf{w}(n+1) \rangle} \right).$$

Hence, we characterize a sufficient condition in the remaining training period to guarantee the directional monotonicity. Moreover, we will show that such a condition can be satisfied when the weight norm $k\mathbf{w}(n)k$ is large enough compared to the current learning rate.

Lemma 5 (A sufficient convergence condition) Under Assumption 1, if $\mathbf{w}(0) \notin \mathbf{0}$, $\theta(0) \notin \pi$, and there exists a $\delta > 0$, s.t.

$$k\mathbf{w}(n+1)k + \langle \bar{\mathbf{w}}(n), \mathbf{w}(n+1) \rangle \geq \frac{(1 + \delta) c_0 \eta_n \cos \theta(n)}{\pi}, \quad \forall n \geq N, \quad (11)$$

then there exist constants $A, B, C > 0$ such that

$$\cos \theta(n) \leq 1 - (1 - \cos \theta(0)) e^{-BS_n^+}. \quad (12)$$

Here $S_n^+ := \sum_{k=0}^{n-1} \frac{\eta_k}{A + \sum_{i=0}^k (\eta_i^2 + C\eta_i)}$, $A = k\mathbf{w}(0)k^2 / c_0^2$, $B = \frac{\delta(1 + \cos \theta(0))}{(1 + \delta)\pi}$, and $C = 0.6 / c_0^2$.

Furthermore, we can show the directional convergence with bounded learning rates as follows. Note that when $k\mathbf{w}(n)k \geq \eta_n c_0 + (1 + \delta) c_0 \eta_n / (2\pi)$, by (i) in Proposition 3, Eq. (11) can be satisfied from

$$k\mathbf{w}(n+1)k + \langle \bar{\mathbf{w}}(n), \mathbf{w}(n+1) \rangle \geq 2(k\mathbf{w}(n)k - \eta_n k \mathbf{r} L(\mathbf{w}(n))k) - (1 + \delta) c_0 \eta_n / \pi.$$

Once $\eta_n \rightarrow \eta_+$, then $\|\mathbf{w}(n)\| \leq R_1 := \eta_+ c_0 + c_0 \eta_+ / \pi$ is sufficient to derive the convergence. We note that $\|\mathbf{w}(n)\| \leq \|\mathbf{v}^> \mathbf{w}(n)\|$, and from (ii) in Proposition 2 and (iii) in Proposition 3, the right term $\|\mathbf{v}^> \mathbf{w}(n)\|$ monotonically increases to infinity. Thus, after finite iterations, the sufficient convergence condition would be satisfied.

Theorem 2 *Under Assumption 1, for a bounded learning rate sequence $\{\eta_n\}_{n=1}^{\infty}$ with $\eta_+ = \limsup \eta_n > 0$, $\cos \theta(n)$ first converges following Eq. (9) until $\cos \theta(n) = 0$ for some $n_0 = 0$. Then there exists an $n_1 \geq n_0$ such that $\|\mathbf{v}^> \mathbf{w}(n_1)\| \leq \eta_+ c_0 + \eta_+ c_0 / \pi$, and $\cos \theta(n)$ converges to 1 from n_1 following Eq. (12).*

Remark 3 *Generally, a β -smooth objective loss has the learning rate constraint ($0 < \eta < \frac{2}{\beta}$) to guarantee convergence [40]. There is no such constraint because the purpose is learning the direction instead of decreasing the loss. Furthermore, we need to underline that the loss still converges. Since for large enough $\|\mathbf{w}(n)\|$, the smoothness coefficient of the objective becomes certainly small and the bounded learning rate is naturally suitable to guarantee convergence.*

Comparison. Our convergence bounds provide a faster and non-asymptotic directional convergence rate $\exp(-O(\frac{\rho}{t}))$ (under constant learning rates, i.e., $\eta_n = \eta$) compared to $O(1/\log^2 t)$ in Soudry et al. [46, Theorem 5]. Additionally, previous results hold for a certain large t and the finite dataset with a positive margin, but we show the behavior in the entire training dynamic. We think that the possible explanation is the benefit of the structured data in Assumption 1. Moreover, the prior technique of proofs mostly employs the decomposition: $\mathbf{w}(t) = \hat{\mathbf{w}} \log t + \rho(t)$ with the max-margin solution $\hat{\mathbf{w}}$ and almost bounded residual term $\rho(t)$. We can implicitly obtain an analogous decomposition, but such a way would lose sight of variation during the early training, such as the decreasing and increasing period of $\|\mathbf{w}(t)\|$. Furthermore, for gradient descent, directional convergence in previous results is built on the loss convergence [22, 24, 38, 39, 46]. Hence they need “small” learning rates with a data-related upper bound. When data are distributed well, we derive the directional convergence directly under implicitly bounded learning rates and then obtain the loss convergence from the directional convergence.

4 Deep Linear Networks

In this section, we extend the results of gradient flow to deep linear networks. For an N -layer linear network $\phi(\mathbf{x}, \mathbf{w}) = W_N \dots W_1 \mathbf{x}$ where $\mathbf{w} := (W_N, \dots, W_1)$, the objective is

$$\min_{\mathbf{w}} L^{(N)}(W_N, \dots, W_1) := \mathbb{E}_{\mathbf{x}} \mathbb{D} \ln \left[1 + e^{-y(\mathbf{x}) W_N \dots W_1 \mathbf{x}} \right].$$

Every such a network represents a linear mapping given by $\mathbf{w}_e = (W_N \dots W_1)^> \in \mathbb{R}^d$:

$$L^{(N)}(W_1, \dots, W_N) = L^{(1)}(\mathbf{w}_e) = \mathbb{E}_{\mathbf{x}} \mathbb{D} \ln \left(1 + e^{-y(\mathbf{x}) \mathbf{w}_e^> \mathbf{x}} \right).$$

4.1 Main results for deep linear networks

A key tool for analyzing the induced flow for \mathbf{w}_e is established in Claim 2 of Arora et al. [3]. Specifically, if the initial balancedness conditions

$$W_{j+1}(0)^> W_{j+1}(0) = W_j(0)^> W_j(0), j = 1, \dots, N-1 \quad (13)$$

hold, then we have the induced gradient flow with $\mathbf{w}_e(t)$:

$$\frac{\partial \mathbf{w}_e(t)}{\partial t} = -\|\mathbf{w}_e(t)\|^2 \frac{2}{N} \left(\frac{dL^{(1)}(\mathbf{w}_e(t))}{d\mathbf{w}} + (N-1) \overline{\mathbf{w}}_e(t) \overline{\mathbf{w}}_e(t)^> \frac{dL^{(1)}(\mathbf{w}_e(t))}{d\mathbf{w}} \right). \quad (14)$$

Such an initialization technique is common for linear networks [2, 3, 7, 18, 19]. Claim 4 in Arora et al. [2] discusses the importance of (nearly) balanced initialization for deep linear networks. For equal-width linear nets, orthogonal initialization is a special case of balanced initialization, and was shown to perform better than Gaussian initialization [19].

Before presenting results of deep linear networks for general N , we first present the result for the simple case $N = 2$:

Proposition 4 Under Assumption 1 and the initial balancedness condition Eq. (13), if $N = 2$, $\mathbf{w}_e(0) \notin \mathbf{0}$ and $\theta(0) \notin \pi$, we obtain the directional convergence as follows:

$$\cos \theta(t) = 1 - \frac{2}{C_1 e^{2c_0 t/\pi} + 1}, \text{ with } C_1 = \frac{1 + \cos \theta(0)}{1 - \cos \theta(0)}.$$

Next, we present a lemma which is an extension of Lemma 1 to the deep linear case: both lemmas present the ‘‘descending-then-ascending’’ pattern of the weight norm.

Lemma 6 Denote $T := \inf \{t : \partial k \mathbf{w}_e(t) k^2 / \partial t < 0\}$. We have $T < 1$, and

$$\frac{\partial k \mathbf{w}_e(t) k^2}{\partial t} \begin{cases} < 0, & \text{if } t < T \\ > 0, & \text{if } t > T \end{cases} \quad (15)$$

The main result for the deep linear net case provides the convergence rate of the gradient flow.

Theorem 3 Under Assumption 1 and the initial balancedness condition Eq. (13), if $N > 2$, $\mathbf{w}_e(0) \notin \mathbf{0}$ and $\theta(0) \notin \pi$, we obtain two-phase convergence below. With the T defined in Lemma 6, we have

$$\cos \theta(t) \begin{cases} 1 - \frac{2}{C_1 (A_1 t / B_1 + 1)^\alpha + 1}, & t \leq T, \\ 1 - \frac{2}{e^{A_2 (t - T) + B_2} + 1}, & t > T. \end{cases}$$

Here $A_1 = (N - 2)c_0$, $B_1 = k \mathbf{w}_e(0) k^{\frac{2}{N} - 1}$, $C_1 = \frac{1 + \cos \theta(0)}{1 - \cos \theta(0)}$, and $A_2 = 2c_0 k \mathbf{w}_e(T) k^{1 - \frac{2}{N}} / \pi$, $B_2 = 2 \ln \left| \tan \frac{\theta(T)}{2} \right|$, and $\alpha = 2 / [(N - 2)\pi]$.

In addition, we have the upper bound that

$$\cos \theta(t) \leq 1 - \frac{2}{e^{F[(0.6t + D)^{N/2} - D^{N/2}] + E} + 1},$$

where $D = k \mathbf{w}_e(0) k^{\frac{2}{N}}$, $E = 2 \ln \left| \frac{\tan \theta(0)}{2} \right|$, and $F = \frac{4c_0}{0.6N\pi}$.

Theorem 3 implies $\cos \theta(t)$ undergoes two stages: polynomial convergence stage and exponential convergence stage. Finally, as for the initialization $\theta(0) = \pi$, we have that $\theta(t) = \pi$, $\forall t \geq 0$, and $\mathbf{w}_e(t) \neq \mathbf{0}$ but never hits the origin by the lower bound in Eq. (17).

Remark 4 We briefly discuss how different items affect the convergence rate of both stages. The convergence speed of the second stage is positively correlated with A_2 , which is positively correlated with c_0 and $k \mathbf{w}_e(T) k^{1 - 2/N}$ (when the number of layers $N \geq 3$). Scaling up the data can increase the expected data norm c_0 , thus increasing A_2 and the convergence speed. It is not clear, though, how to increase $k \mathbf{w}_e(T) k$ since it depends on an unknown transition time T . The convergence speed of the first stage is positively correlated with A_1 and $1/B_1$. We skip the discussions of this stage.

Remark 5 Since we need to cover the worse case during optimization, our bound may seem loose in two possible cases. First, when $t \leq T$, we use the lower bound in Eq. (17) to capture the decreasing period of $k \mathbf{w}_e(t) k$. Actually, $k \mathbf{w}_e(t) k$ may descend to a certain value that is higher than the lower bound in Eq. (17) before ascending. Second, when $t > T$, note that $\cos \theta(t) \neq 1$ except $\mathbf{w}_e(0) = k \mathbf{v}$ for $k > 0$ by Theorem 3. We can guarantee $k \mathbf{w}_e(t) k \neq 1$ because $\mathbf{v}^T \mathbf{w}_e(t)$ is increasing after some time (but not always). Hence, the convergence could be faster when $k \mathbf{w}_e(t) k$ increases much, but we only treat such a scheme lower bounded by $k \mathbf{w}_e(T) k$.

Comparison. Our global convergence rate results for deep linear networks neither require small training loss [37, 21, 23] nor presume convergence of loss and weight direction [17, 38]. Moreover, our results do not require overparameterization [9, 11, 19]. However, we need the exact initial balancedness condition [3]. Previous works [37, 21, 51] mainly provided asymptotic directional convergence to the maximum margin solution given zero classification error at the beginning. We could obtain an explicit global directional convergence including the early training dynamic. Our directional rates could be linear when the induced weight norm begins increasing, which is faster than the polynomial convergence rate of the surrogate loss function obtained in [37, Theorem 4.3]. However, our results heavily depend on the benign training data assumption and the linear net structure compared to Lyu and Li [37], Yun et al. [51].

4.2 Technical lemmas for analyzing deep linear networks

We present a series of technical results that will be used for proving Lemma 6 and Theorem 3. Some are extensions of those for logistic regression, and some are new for deep nets.

Lemma 7 *Under Assumption 1 and the initial balancedness condition Eq. (13), if $\mathbf{w}_e(t) \notin \mathbf{0}$, then*

$$\frac{\partial \cos \theta(\mathbf{w}_e(t), \mathbf{v})}{\partial t} = \frac{c_0 \sin^2 \theta(\mathbf{w}_e(t), \mathbf{v})}{\pi} k\mathbf{w}_e(t)k^{1-\frac{2}{N}}. \quad (16)$$

The main difference from the logistic regression case is that the dependence of weight norm $k\mathbf{w}_e(t)k$ is reversed. Larger $k\mathbf{w}_e(t)k$ gives faster convergence for the deep linear networks when $N \geq 3$, while no influence for $N = 2$, and for $N = 1$ it is opposite (i.e. larger $k\mathbf{w}_e(t)k$ gives slower convergence).

We then provide a few properties of the weight norm during training.

Lemma 8 *Under Assumption 1 and the initial balancedness condition Eq. (13) with $\mathbf{w}_e(0) \notin \mathbf{0}$, we have the following two properties:*

(i) *Suppose $N > 2$. For any $t \geq 0$, we have*

$$\left(k\mathbf{w}_e(0)k^{\frac{2}{N}} + 0.6t \right)^{\frac{N}{2}} - k\mathbf{w}_e(t)k \left(k\mathbf{w}_e(0)k^{\frac{2}{N}-1} + (N-2)c_0t \right)^{\frac{N}{N-2}} > 0. \quad (17)$$

(ii) *Suppose $\partial k\mathbf{w}_e(t_0)k^2 / \partial t = 0$ with $\mathbf{w}_e(t_0) \notin \mathbf{0}$ for some $t_0 \geq 0$. Then $\partial g(\mathbf{w}_e(t_0)) / \partial t > 0$.*

We briefly explain how to use Lemma 8 to prove Lemma 6, which consists of two properties: Eq. (15) and $T < 1$. First, Eq. (15) is an immediate consequence of Part (ii) of Lemma 8, as well as a calculus fact Lemma 10 (see Appendix). This proof approach is similar to that for the logistic regression case. Second, $T < 1$ requires a different proof approach from the logistic regression case: the latter is due to unbounded weight norm and the fact that zero-norm weight is non-stationary, but for deep linear case the latter does not hold (the origin is a saddle point). Instead, we rely on Part (i) of Lemma 8 to prove $T < 1$ by contradiction. More specifically, assume $T = 1$, i.e., $k\mathbf{w}_e(t)k$ is decreasing for all t , then $k\mathbf{w}_e(t)k$ converges to a finite value. We will use Part (i) of Lemma 8 and Part (iv) of Lemma 3 to derive a contradiction. See details in Appendix B.2.

4.3 Proof of Theorem 3

Proof: Using Lemma 7 and the lower bound of $k\mathbf{w}_e(t)k$ in Lemma 8, we obtain

$$\frac{1}{2} \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} - \frac{1}{2} \ln \frac{1 + \cos \theta(0)}{1 - \cos \theta(0)} - \frac{1}{(N-2)\pi} \ln \frac{A_1 t + B_1}{B_1}.$$

This implies

$$\cos \theta(t) \geq 1 - \frac{2}{C_1(A_1 t / B_1 + 1)^\alpha + 1}, \quad \forall t \geq 0. \quad (18)$$

Notice that (18) holds for the whole training period, thus surely holds for $t \leq T$.

Next, we will show a stronger convergence rate for the period $t > T$, i.e., the second part of the theorem. According to Lemma 6 we have

$$\frac{\partial k\mathbf{w}_e(t)k^{\frac{2}{N}}}{\partial t} \leq 0, \quad \forall t \geq T.$$

Hence $k\mathbf{w}_e(t)k \geq k\mathbf{w}_e(T)k, \forall t \geq T$. Therefore,

$$\begin{aligned} \frac{1}{2} \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} &\geq \frac{1}{2} \ln \frac{1 + \cos \theta(T)}{1 - \cos \theta(T)} - \frac{c_0}{\pi} k\mathbf{w}_e(T)k^{1-\frac{2}{N}}(t-T) \\ &=) \cos \theta(t) \geq 1 - \frac{2}{e^{A_2(t-T)+B_2} + 1}, \quad \forall t \geq T. \end{aligned}$$

For the third part, using the upper bound of $k\mathbf{w}_e(t)k$ in Lemma 8, we obtain

$$\frac{1}{2} \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} - \frac{1}{2} \ln \frac{1 + \cos \theta(0)}{1 - \cos \theta(0)} \leq \frac{2c_0}{0.6N\pi} \left((0.6t + D)^{\frac{N}{2}} - D^{\frac{N}{2}} \right).$$

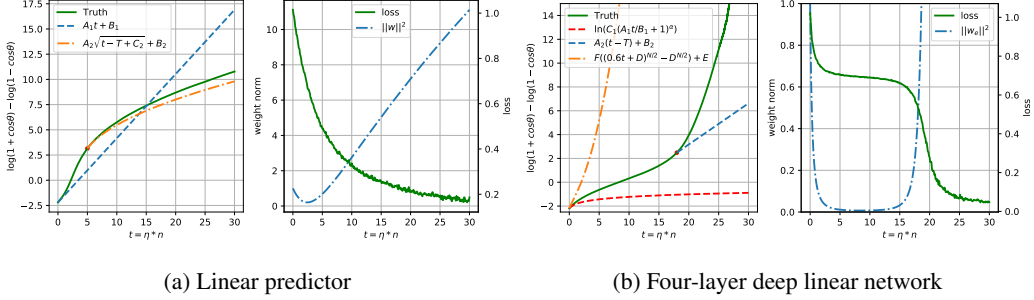


Figure 1: Simulation of (deep) linear network with $\mathbf{x} \sim U(S^1)$, $\mathbf{v} = (0, 1)^\top$. (a): Linear predictor. We show our lower bounds in Theorem 1 at $n = 0$ and $n = 5000$. (b): Four-layer deep linear network. We also show our lower and upper bounds in Theorem 3 at $n = 0$ and $n = 18000$. In each experiment, we plot angle variation, loss and weight norm in sequence.

This implies

$$\cos \theta(t) \geq 1 - \frac{2}{e^{F[(0.6t+D)^{N/2} - D^{N/2}] + E} + 1}.$$

5 Experiments

In this section we conduct experiments to verify our theoretical analyses. We consider training a linear classifier and a 4-layer linear network under the logit loss as our theorems show. We construct simple dataset with $\mathbf{x} \sim U(S^1)$ and $y(\mathbf{x}) = \text{sgn}(\mathbf{v}^\top \mathbf{x})$ with $\mathbf{v} = (0, 1)^\top$. We use common stochastic gradient descent (SGD) with the batch size 1000 and the constant small learning rate 10^{-3} . Moreover, we choose an initial value $\mathbf{w}(0) = \mathbf{w}_e(0) = (0.6, 0.8)^\top$. In the deep linear network, we set $W_N(0) = \mathbf{u}_N^\top$, $W_i(0) = \mathbf{u}_{i+1} \mathbf{u}_i^\top$ with $k\mathbf{u}_i k = 1, i = 2, \dots, N$ and $\mathbf{u}_1 = \mathbf{w}_e(0)$ to satisfy the balancedness conditions Eq. (13). The results are shown in Figure 1.

Figure 1a shows the optimization period for linear classifiers. We also plot the convergence bounds obtained in Theorems 1. Although we do not give convergence for SGD, our bounds in gradient flow still roughly match the directional convergence in practice, and the weight norm $k\mathbf{w}(n)k$ indeed goes through a decreasing and increasing period. Figure 1b depicts the dynamic for the deep linear networks. We also plot three convergence bounds in Theorems 3, which roughly match the actual behavior. The weight norm $k\mathbf{w}_e(n)k$ still goes through the decreasing and increasing period, and we could observe a distinct stuck period when $5 \leq \eta n \leq 10$ as the lower bound shown in Lemma 8. Moreover, the loss decreases slowly when $k\mathbf{w}(n)k$ is small, but $\cos \theta(t)$ still has considerable growth. Thus previous loss-based analysis may neglect potential variation of the accuracy we most concern. Furthermore, we can see a more rigorous understanding of $k\mathbf{w}(n)k$ may give more precise rates in the late training period ($t \geq 20$).

6 Conclusion

In this work, we have studied the training dynamic of (deep) linear networks in binary classification tasks. Specifically, we focus on the gradient flow and gradient descent methods under population loss. We have proved exact directional convergence for (deep) linear networks in the entire training process on spherically symmetric data. Moreover, we have characterized the descent and ascent behavior of the induced weight norm theoretically. Such a phenomenon also is observed in numerical experiments. We hope that our limited view of directional convergence would bring a better understanding of (linear) neural networks.

Acknowledgments and Disclosure of Funding

Lin and Zhang have been supported by the National Key Research and Development Project of China (No. 2018AAA0101004) and Beijing Natural Science Foundation (Z190001).

References

- [1] Zeyuan Allen-Zhu, Yuanzhi Li, and Zhao Song. A convergence theory for deep learning via over-parameterization. In *International Conference on Machine Learning*, pages 242–252. PMLR, 2019.
- [2] Sanjeev Arora, Nadav Cohen, Noah Golowich, and Wei Hu. A convergence analysis of gradient descent for deep linear neural networks. In *International Conference on Learning Representations*, 2018.
- [3] Sanjeev Arora, Nadav Cohen, and Elad Hazan. On the optimization of deep networks: Implicit acceleration by overparameterization. In *International Conference on Machine Learning*, pages 244–253. PMLR, 2018.
- [4] Sanjeev Arora, Simon Du, Wei Hu, Zhiyuan Li, and Ruosong Wang. Fine-grained analysis of optimization and generalization for overparameterized two-layer neural networks. In *International Conference on Machine Learning*, pages 322–332. PMLR, 2019.
- [5] Francis Bach and Lenaic Chizat. Gradient descent on infinitely wide neural networks: Global convergence and generalization. *arXiv preprint arXiv:2110.08084*, 2021.
- [6] Bubacarr Bah, Holger Rauhut, Ulrich Terstiege, and Michael Westdickenberg. Learning deep linear neural networks: Riemannian gradient flows and convergence to global minimizers. *Information and Inference: A Journal of the IMA*, 02 2021. ISSN 2049-8772. doi: 10.1093/imaia/iaaa039. URL <https://doi.org/10.1093/imaia/iaaa039>. iaaa039.
- [7] Peter Bartlett, Dave Helmbold, and Philip Long. Gradient descent with identity initialization efficiently learns positive definite linear transformations by deep residual networks. In *International conference on machine learning*, pages 521–530. PMLR, 2018.
- [8] Alon Brutzkus and Amir Globerson. Globally optimal gradient descent for a convnet with gaussian inputs. In *International conference on machine learning*, pages 605–614. PMLR, 2017.
- [9] Lenaic Chizat and Francis Bach. Implicit bias of gradient descent for wide two-layer neural networks trained with the logistic loss. In *Conference on Learning Theory*, pages 1305–1338. PMLR, 2020.
- [10] Tian Ding, Dawei Li, and Ruoyu Sun. Sub-optimal local minima exist for almost all over-parameterized neural networks. *arXiv preprint arXiv:1911.01413*, 2019.
- [11] Simon Du and Wei Hu. Width provably matters in optimization for deep linear neural networks. In *International Conference on Machine Learning*, pages 1655–1664. PMLR, 2019.
- [12] Simon Du, Jason Lee, Haochuan Li, Liwei Wang, and Xiyu Zhai. Gradient descent finds global minima of deep neural networks. In *International Conference on Machine Learning*, pages 1675–1685. PMLR, 2019.
- [13] Simon S Du, Xiyu Zhai, Barnabas Poczos, and Aarti Singh. Gradient descent provably optimizes over-parameterized neural networks. In *International Conference on Learning Representations*, 2018.
- [14] Armin Eftekhari. Training linear neural networks: Non-local convergence and complexity results. In *International Conference on Machine Learning*, pages 2836–2847. PMLR, 2020.
- [15] Jonathan Frankle, David J. Schwab, and Ari S. Morcos. The early phase of neural network training. In *International Conference on Learning Representations*, 2020. URL <https://openreview.net/forum?id=Hkl1iRNFwS>.

- [16] Suriya Gunasekar, Jason Lee, Daniel Soudry, and Nathan Srebro. Characterizing implicit bias in terms of optimization geometry. In *International Conference on Machine Learning*, pages 1832–1841. PMLR, 2018.
- [17] Suriya Gunasekar, Jason D Lee, Daniel Soudry, and Nati Srebro. Implicit bias of gradient descent on linear convolutional networks. In *Advances in Neural Information Processing Systems*, pages 9461–9471, 2018.
- [18] Moritz Hardt and Tengyu Ma. Identity matters in deep learning. In *International Conference on Learning Representations*, 2017.
- [19] Wei Hu, Lechao Xiao, and Jeffrey Pennington. Provable benefit of orthogonal initialization in optimizing deep linear networks. In *International Conference on Learning Representations*, 2019.
- [20] Arthur Jacot, Franck Gabriel, and Clément Hongler. Neural tangent kernel: Convergence and generalization in neural networks. In *Advances in neural information processing systems*, pages 8571–8580, 2018.
- [21] Ziwei Ji and Matus Telgarsky. Gradient descent aligns the layers of deep linear networks. In *7th International Conference on Learning Representations, ICLR 2019*, 2019.
- [22] Ziwei Ji and Matus Telgarsky. The implicit bias of gradient descent on nonseparable data. In *Conference on Learning Theory*, pages 1772–1798. PMLR, 2019.
- [23] Ziwei Ji and Matus Telgarsky. Directional convergence and alignment in deep learning. *Advances in Neural Information Processing Systems*, 33, 2020.
- [24] Ziwei Ji and Matus Telgarsky. Characterizing the implicit bias via a primal-dual analysis. In *Algorithmic Learning Theory*, pages 772–804. PMLR, 2021.
- [25] Ziwei Ji, Miroslav Dudík, Robert E Schapire, and Matus Telgarsky. Gradient descent follows the regularization path for general losses. In *Conference on Learning Theory*, pages 2109–2136. PMLR, 2020.
- [26] Kenji Kawaguchi. Deep learning without poor local minima. In *Advances in neural information processing systems*, pages 586–594, 2016.
- [27] Thomas Laurent and James Brecht. Deep linear networks with arbitrary loss: All local minima are global. In *International Conference on Machine Learning*, pages 2908–2913, 2018.
- [28] Chris Junchi Li, Mengdi Wang, Han Liu, and Tong Zhang. Diffusion approximations for online principal component estimation and global convergence. *Advances in Neural Information Processing Systems*, page 646, 2017.
- [29] Chris Junchi Li, Mengdi Wang, Han Liu, and Tong Zhang. Near-optimal stochastic approximation for online principal component estimation. *Mathematical Programming*, 167(1):75–97, 2018.
- [30] Dawei Li, Tian Ding, and Ruoyu Sun. On the benefit of width for neural networks: Disappearance of bad basins, 2021.
- [31] Yuanzhi Li and Yingyu Liang. Learning overparameterized neural networks via stochastic gradient descent on structured data. In *Advances in Neural Information Processing Systems*, pages 8157–8166, 2018.
- [32] Yuanzhi Li, Tengyu Ma, and Hongyang R Zhang. Learning over-parametrized two-layer neural networks beyond ntk. In *Conference on Learning Theory*, pages 2613–2682. PMLR, 2020.
- [33] Shiyu Liang, Ruoyu Sun, Jason D Lee, and R Srikant. Adding one neuron can eliminate all bad local minima. In *Advances in Neural Information Processing Systems*, pages 4355–4365, 2018.

- [34] Shiyu Liang, Ruoyu Sun, Yixuan Li, and Rayadurgam Srikant. Understanding the loss surface of neural networks for binary classification. In Jennifer Dy and Andreas Krause, editors, *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 2835–2843. PMLR, 10–15 Jul 2018. URL <https://proceedings.mlr.press/v80/liang18a.html>.
- [35] Shiyu Liang, Ruoyu Sun, and R. Srikant. Revisiting landscape analysis in deep neural networks: Eliminating decreasing paths to infinity, 2019.
- [36] Haihao Lu and Kenji Kawaguchi. Depth creates no bad local minima. *arXiv preprint arXiv:1702.08580*, 2017.
- [37] Kaifeng Lyu and Jian Li. Gradient descent maximizes the margin of homogeneous neural networks. In *International Conference on Learning Representations*, 2019.
- [38] Mor Shpigel Nacson, Jason Lee, Suriya Gunasekar, Pedro Henrique Pamlona Savarese, Nathan Srebro, and Daniel Soudry. Convergence of gradient descent on separable data. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 3420–3428. PMLR, 2019.
- [39] Mor Shpigel Nacson, Nathan Srebro, and Daniel Soudry. Stochastic gradient descent on separable data: Exact convergence with a fixed learning rate. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 3051–3059. PMLR, 2019.
- [40] Yurii Nesterov. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media, 2013.
- [41] Quynh Nguyen, Mahesh Chandra Mukkamala, and Matthias Hein. On the loss landscape of a class of deep neural networks with no bad local valleys. *arXiv preprint arXiv:1809.10749*, 2018.
- [42] Maher Nouiehed and Meisam Razaviyayn. Learning deep models: Critical points and local openness. *arXiv preprint arXiv:1803.02968*, 2018.
- [43] Mary Phuong and Christoph Lampert. Towards understanding knowledge distillation. In *International Conference on Machine Learning*, pages 5142–5151. PMLR, 2019.
- [44] Itay Safran and Ohad Shamir. Spurious local minima are common in two-layer relu neural networks. *arXiv preprint arXiv:1712.08968*, 2017.
- [45] Itay Safran and Ohad Shamir. Spurious local minima are common in two-layer relu neural networks. In *International Conference on Machine Learning*, pages 4433–4441. PMLR, 2018.
- [46] Daniel Soudry, Elad Hoffer, Mor Shpigel Nacson, Suriya Gunasekar, and Nathan Srebro. The implicit bias of gradient descent on separable data. *The Journal of Machine Learning Research*, 19(1):2822–2878, 2018.
- [47] Ruo-Yu Sun. Optimization for deep learning: An overview. *Journal of the Operations Research Society of China*, pages 1–46, 2020.
- [48] Ruoyu Sun, Dawei Li, Shiyu Liang, Tian Ding, and Rayadurgam Srikant. The global landscape of neural networks: An overview. *IEEE Signal Processing Magazine*, 37(5):95–108, 2020.
- [49] Luca Venturi, Afonso Bandeira, and Joan Bruna. Spurious valleys in two-layer neural network optimization landscapes. *arXiv preprint arXiv:1802.06384*, 2018.
- [50] Gilad Yehudai and Ohad Shamir. Learning a single neuron with gradient methods. In *Conference on Learning Theory*, pages 3756–3786. PMLR, 2020.
- [51] Chulhee Yun, Shankar Krishnan, and Hossein Mobahi. A unifying view on implicit bias in training linear neural networks. In *International Conference on Learning Representations*, 2021. URL <https://openreview.net/forum?id=ZsZM-4iMQkH>.
- [52] Li Zhang. Depth creates no more spurious local minima. *arXiv preprint arXiv:1901.09827*, 2019.

- [53] Kai Zhong, Zhao Song, Prateek Jain, Peter L Bartlett, and Inderjit S Dhillon. Recovery guarantees for one-hidden-layer neural networks. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 4140–4149. JMLR. org, 2017.
- [54] Mo Zhou, Rong Ge, and Chi Jin. A local convergence theory for mildly over-parameterized two-layer neural network. In *Conference on Learning Theory*, pages 4577–4632. PMLR, 2021.

A Missing Proofs in Section 3

A.1 Proof of Proposition 1

Proof: $L(\mathbf{w})$ is Lipschitz continuous following from

$$\| \nabla L(\mathbf{w}) \| \leq \mathbb{E}_{\mathbf{x}} \left\| \frac{y(\mathbf{x})}{1 + e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \mathbf{x} \right\| \leq \mathbb{E}_{\mathbf{x}} \|\mathbf{x}\| \sqrt{\mathbb{E}_{\mathbf{x}} \|\mathbf{x}\|^2} = \sqrt{\text{tr}(\Sigma)}.$$

The convexity of $L(\mathbf{w})$ is proved by

$$r^2 L(\mathbf{w}) = \mathbb{E}_{\mathbf{x}} \frac{1}{1 + e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \frac{e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}}{1 + e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \mathbf{x}\mathbf{x}^\top \succeq \mathbf{0}.$$

Moreover, for any $\mathbf{u} \succeq \mathbb{R}^d$ with $\|\mathbf{u}\| = 1$,

$$\mathbf{u}^\top r^2 L(\mathbf{w}) \mathbf{u} = \mathbb{E}_{\mathbf{x}} \frac{1}{1 + e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \frac{e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}}{1 + e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \mathbf{x}\mathbf{x}^\top \mathbf{u}\mathbf{u}^\top \geq \frac{1}{4} \mathbb{E} \|\mathbf{x}\|^2 \geq \frac{1}{4} \lambda_{\max}(\Sigma).$$

However, note that

$$\lim_{k \rightarrow \infty} \frac{1}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \frac{e^{k\mathbf{v}^\top \mathbf{x}_j}}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \mathbf{x}_j \mathbf{x}_j^\top \leq \frac{\|\mathbf{x}_j\|^2}{4},$$

then from dominated convergence theorem, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbf{x}} \frac{1}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \frac{e^{k\mathbf{v}^\top \mathbf{x}_j}}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \mathbf{x}_j \mathbf{x}_j^\top &= \mathbb{E}_{\mathbf{x}} \lim_{k \rightarrow \infty} \frac{1}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \frac{e^{k\mathbf{v}^\top \mathbf{x}_j}}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \mathbf{x}_j \mathbf{x}_j^\top \\ &= \mathbb{E}_{\mathbf{x}} \|\mathbf{x}_j\|^2 \mathbb{1}_{\mathbf{v}^\top \mathbf{x}_j = 0} = 0, \end{aligned} \quad (19)$$

where the final equality uses the assumption $P(\mathbf{v}^\top \mathbf{x} = 0) = 0$. Then we get

$$\limsup_{k \rightarrow \infty, \mathbf{w} = k\mathbf{v}} r^2 L(\mathbf{w}) \preceq \limsup_{k \rightarrow \infty} \mathbb{E}_{\mathbf{x}} \frac{1}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \frac{e^{k\mathbf{v}^\top \mathbf{x}_j}}{1 + e^{k\mathbf{v}^\top \mathbf{x}_j}} \mathbf{x}_j \mathbf{x}_j^\top \stackrel{(19)}{=} \mathbf{0}.$$

Therefore

$$\lim_{\substack{\mathbf{w} = k\mathbf{v} \\ k \rightarrow \infty}} r^2 L(\mathbf{w}) = \mathbf{0}.$$

Thus $L(\mathbf{w})$ is not strongly convex.

A.2 Proof of Proposition 2

Proof: Proof of (i): Suppose $\|\mathbf{w}\| \leq M$ for some positive constant M , and notice that

$$\mathbf{v}^\top \nabla L(\mathbf{w}) = \mathbb{E}_{\mathbf{x}} \frac{j\mathbf{v}^\top \mathbf{x}_j}{1 + e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \succeq \mathbf{0}.$$

More specifically,

$$\mathbf{v}^\top \nabla L(\mathbf{w}) \succeq \mathbb{E}_{\mathbf{x}} \frac{j\mathbf{v}^\top \mathbf{x}_j}{1 + e^{k\mathbf{w}^\top \mathbf{x}_j}} \frac{\mathbb{E}_{\mathbf{x}} [j\mathbf{v}^\top \mathbf{x}_j \mathbb{1}_{\|\mathbf{x}_j\| \leq R}]}{1 + e^{MR}} \frac{r P(\|\mathbf{x}_j\| \leq R, j\mathbf{v}^\top \mathbf{x}_j \geq r)}{1 + e^{MR}}.$$

Let $\epsilon := P(j\mathbf{v}^\top \mathbf{x}_j > 0)$, then $\epsilon > 0$ from assumption $P(\mathbf{v}^\top \mathbf{x} = 0) < 1$. There exist $R > r > 0$, such that $P(\|\mathbf{x}_j\| \leq R, j\mathbf{v}^\top \mathbf{x}_j \geq r) \geq \epsilon/2$, then $\mathbf{v}^\top \nabla L(\mathbf{w}(t)) \succeq 0.5r\epsilon/(1 + e^{MR}) > \mathbf{0}$.

Proof of (ii): According to (i), we have $\partial(\mathbf{v}^\top \mathbf{w}(t))/\partial t \succeq \mathbf{0}$ and $\mathbf{v}^\top (\mathbf{w}(n+1) - \mathbf{w}(n)) \succeq \mathbf{0}$, showing that $\mathbf{v}^\top \mathbf{w}(t)$ and $\mathbf{v}^\top \mathbf{w}(n)$ are increasing.

Proof of (iii): Assume the contrary that $\|\mathbf{w}(t)\| \leq M$, $\delta t \leq 0$ for certain M . Then $\mathbf{v}^\top \mathbf{w}(t) \leq \|\mathbf{w}(t)\| \leq M$, $\delta t \leq 0$. Hence, $\mathbf{v}^\top \mathbf{w}(t)$ converges from (ii), showing that $\lim_{t \rightarrow \infty} \mathbf{v}^\top \nabla L(\mathbf{w}(t)) = \mathbf{0}$. Using the previous argument again, we obtain a contradiction. Thus the assumption does not hold, i.e., $\|\nabla L(\mathbf{w}(t))\|$ is unbounded.

Proof of (iv): Assume the contrary that $\|\mathbf{w}(n)\| \leq M$, $\delta n \leq 0$ for certain M . Then $\mathbf{v}^\top \mathbf{w}(n) \leq \|\mathbf{w}(n)\| \leq M$, $\delta n \leq 0$. Hence, $\mathbf{v}^\top \mathbf{w}(n)$ converges from (ii), showing that $\lim_{n \rightarrow \infty} \eta_n \mathbf{v}^\top \nabla L(\mathbf{w}(n)) = \mathbf{0}$. Since $\eta_n \geq \eta > 0$, then $\lim_{n \rightarrow \infty} \mathbf{v}^\top \nabla L(\mathbf{w}(n)) = \mathbf{0}$, remaining the same argument as the gradient flow case.

A.3 Proof of Proposition 3

Proposition 5 (Restatement of Proposition 3.) *Under Assumption 1, we have: (i) $k\Gamma L(\mathbf{w})k$ $c_0 = \mathbb{E}_{\mathbf{x} \sim D_2} k\mathbf{x}k$; (ii) $\Gamma L(a\mathbf{v}) = \mathbb{E}_t \mathbb{D}_{\mathbf{v}} \frac{jtj}{1+e^{ajt}} \mathbf{v}$; (iii) $\lim_{t \rightarrow \infty} \mathbf{v}^\top \mathbf{w}(t) = +1$. (iv) If the learning rates are lower bounded, i.e., $\eta_n \geq \eta > 0$, then $\lim_{n \rightarrow \infty} \mathbf{v}^\top \mathbf{w}(n) = +1$.*

Proof: Recall that $D_{\mathbf{w}, \mathbf{v}}$ denotes the marginal distribution of $\mathbf{x} = (x_1, \dots, x_d) \sim D$ on the subspace spanned by two linearly independent vectors \mathbf{w}, \mathbf{v} (a distribution over \mathbb{R}^2),

Since D is spherically symmetric, and $\frac{y(\mathbf{x})\mathbf{x}}{1+e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} = \frac{\text{sgn}(\mathbf{x}^\top \mathbf{v})}{1+e^{\text{sgn}(\mathbf{x}^\top \mathbf{v})\mathbf{w}^\top \mathbf{x}}}$ is only decided by $\mathbf{v}^\top \mathbf{x}$ and $\mathbf{w}^\top \mathbf{x}$, we could marginalize the distribution to $\text{span}\{\mathbf{w}, \mathbf{v}\}$, leading to

$$\Gamma L(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim D} \frac{y(\mathbf{x})\mathbf{x}}{1+e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} = \mathbb{E}_{\mathbf{x} \sim D_{\mathbf{w}, \mathbf{v}}} \frac{y(\mathbf{x})\mathbf{x}}{1+e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \in \text{span}\{\mathbf{w}, \mathbf{v}\}. \quad (20)$$

Proof of (i):

$$k\Gamma L(\mathbf{w})k \stackrel{(20)}{=} \left\| \mathbb{E}_{\mathbf{x} \sim D_{\mathbf{w}, \mathbf{v}}} \frac{y(\mathbf{x})\mathbf{x}}{1+e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \right\| = \mathbb{E}_{\mathbf{x} \sim D_{\mathbf{w}, \mathbf{v}}} k\mathbf{x}k = \mathbb{E}_{\mathbf{x} \sim D_2} k\mathbf{x}k = c_0.$$

The second equality uses the property that D is spherically symmetric, then $D_{\mathbf{w}, \mathbf{v}}$ is identical distribution for any \mathbf{w}, \mathbf{v} . We choose the first two coordinate directions for example.

Proof of (ii): when $\mathbf{w} = a\mathbf{v}$, $D_{\mathbf{w}, \mathbf{v}}$ reduce to $D_{\mathbf{v}}$. Thus, we only need to consider the expectation on $\mathbf{x} : \mathbf{x} = t\mathbf{v}, t \in \mathbb{R}$, which leads to $y(\mathbf{x})\mathbf{x} = \text{sgn}(\mathbf{v}^\top \mathbf{x})\mathbf{x} = jtj\mathbf{v}$, and

$$\Gamma L(a\mathbf{v}) \stackrel{(20)}{=} \mathbb{E}_{\mathbf{x} \sim D_{\mathbf{w}, \mathbf{v}}} \frac{y(\mathbf{x})\mathbf{x}}{1+e^{y(\mathbf{x})a\mathbf{v}^\top \mathbf{x}}} = \mathbb{E}_t \mathbb{D}_{\mathbf{v}} \frac{jtj}{1+e^{ajt}} \mathbf{v}.$$

Proof of (iii) and (iv): Finally, letting $\mathbf{w} = \alpha\mathbf{v} + \beta\mathbf{z}$, where $\mathbf{z} \perp \mathbf{v}$ and $k\mathbf{z}k = 1$, by dominated convergence theorem, we have

$$\lim_{\beta \rightarrow \infty} \mathbf{v}^\top \Gamma L(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim D} \lim_{\beta \rightarrow \infty} \frac{j\mathbf{v}^\top \mathbf{x}j}{1+e^{\beta y(\mathbf{x})\mathbf{z}^\top \mathbf{x} + \alpha j\mathbf{v}^\top \mathbf{x}j}} = \mathbb{E}_{(\mathbf{z}^\top \mathbf{x})} \frac{j\mathbf{v}^\top \mathbf{x}j}{1+e^{\alpha j\mathbf{v}^\top \mathbf{x}j}} = \frac{c_0}{\pi}, \quad (21)$$

where the last equality uses the distribution assumption.

Now we denote $\mathbf{w}(t) = \alpha(t)\mathbf{v} + \beta(t)\mathbf{z}(t)$. Assume the contrary, that $\mathbf{v}^\top \mathbf{w}(t) = \alpha(t)$ is bounded. Then from (iii) in Proposition 2, $k\mathbf{w}(t)k$ is unbounded, which implies $\beta(t)$ is unbounded. Since we have assumed $\alpha(t)$ is bounded, and (ii) in Proposition 2 implies $\alpha(t)$ is increasing, we obtain that $\alpha(t)$ converges to a certain finite value. This further implies that $\mathbf{v}^\top \Gamma L(\mathbf{w}(t)) \rightarrow 0$, which is a contradiction to (21). Thus the earlier assumption does not hold, i.e., $\lim_{t \rightarrow \infty} \mathbf{v}^\top \mathbf{w}(t) = +1$.

The same argument holds for $\mathbf{w}(n)$ if the learning rates are lower bounded.

A.4 Proof of Lemma 2

Proof: As mentioned earlier, we have

$$\Gamma L(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim D} \frac{y(\mathbf{x})\mathbf{x}}{1+e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \stackrel{(20)}{=} \mathbb{E}_{\mathbf{x} \sim D_{\mathbf{w}, \mathbf{v}}} \frac{y(\mathbf{x})\mathbf{x}}{1+e^{y(\mathbf{x})\mathbf{w}^\top \mathbf{x}}} \in \text{span}\{\mathbf{w}, \mathbf{v}\}.$$

Thus we only need to restrict our discussion to the two-dimensional linear space $\text{span}\{\mathbf{w}, \mathbf{v}\}$. With a little abuse of notation, we assume $\mathbf{w}, \mathbf{v} \in \mathbb{R}^2$ from now and use \mathbf{w}, \mathbf{v} them as the representations of \mathbf{w}, \mathbf{v} in the subspace $\text{span}\{\mathbf{w}, \mathbf{v}\} \subset \mathbb{R}^2$. Similarly, we assume $\mathbf{x} \in \mathbb{R}^2$ from now on as well, and assume $D_{\mathbf{w}, \mathbf{v}}$ is a distribution on \mathbb{R}^2 (corresponding to the original distribution on $\text{span}\{\mathbf{w}, \mathbf{v}\}$). Then we have

$$\rho := (\mathbf{v} - (\overline{\mathbf{w}}^\top \mathbf{v}) \overline{\mathbf{w}})^\top \Gamma L(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim D_{\mathbf{w}, \mathbf{v}}} \frac{j\mathbf{v}^\top \mathbf{x}j}{1+e^{\text{sgn}(\mathbf{v}^\top \mathbf{x})\mathbf{w}^\top \mathbf{x}}} \frac{\text{sgn}(\mathbf{v}^\top \mathbf{x}) (\overline{\mathbf{w}}^\top \mathbf{v}) (\overline{\mathbf{w}}^\top \mathbf{x})}{1+e^{\text{sgn}(\mathbf{v}^\top \mathbf{x})\mathbf{w}^\top \mathbf{x}}}. \quad (22)$$

²Rigorously speaking, we would need to use different notations such as $\hat{\mathbf{w}} \in \mathbb{R}^2$ to represent the coordinate of \mathbf{w} in $\text{span}\{\mathbf{w}, \mathbf{v}\}$. Anyhow, for simplicity we just reuse the notation \mathbf{w}, \mathbf{v} .

Additionally, the expression above is invariant to the rotation of the coordinate frame on $\text{span}\{\bar{\mathbf{w}}, \mathbf{v}\}$ due to the spherical symmetry of $D_{\mathbf{w}, \mathbf{v}}$, so we can assume without loss of generality that $\bar{\mathbf{w}} = (1, 0)^\top$, $\mathbf{v} = (v_1, v_2)^\top$. Then the numerator of (22) (for a given \mathbf{x} inside the expectation) can be simplified to

$$j\mathbf{v}^\top \mathbf{x} j \quad \text{sgn}(\mathbf{v}^\top \mathbf{x}) (\bar{\mathbf{w}}^\top \mathbf{v}) (\bar{\mathbf{w}}^\top \mathbf{x}) = \text{sgn}(\mathbf{v}^\top \mathbf{x}) (v_1 x_1 + v_2 x_2) \quad \text{sgn}(\mathbf{v}^\top \mathbf{x}) v_1 x_1 = \text{sgn}(\mathbf{v}^\top \mathbf{x}) v_2 x_2.$$

Then (22) can be simplified to

$$\rho = \mathbb{E}_{\mathbf{x}} \int_{D_{\mathbf{w}, \mathbf{v}}} \frac{j\mathbf{v}^\top \mathbf{x} j \quad \text{sgn}(\mathbf{v}^\top \mathbf{x}) (\bar{\mathbf{w}}^\top \mathbf{v}) (\bar{\mathbf{w}}^\top \mathbf{x})}{1 + e^{\text{sgn}(\mathbf{v}^\top \mathbf{x}) \mathbf{w}^\top \mathbf{x}}} = \mathbb{E}_{\mathbf{x}} \int_{D_{\mathbf{w}, \mathbf{v}}} \frac{\text{sgn}(\mathbf{v}^\top \mathbf{x}) (v_2 x_2)}{1 + e^{\text{sgn}(\mathbf{v}^\top \mathbf{x}) k \mathbf{w} k x_1}} = \mathbb{E}_{\mathbf{x}} \int_{D_{\mathbf{w}, \mathbf{v}}} g(\mathbf{x})$$

where $g(\mathbf{x}) = \frac{\text{sgn}(\mathbf{v}^\top \mathbf{x}) (v_2 x_2)}{1 + e^{\text{sgn}(\mathbf{v}^\top \mathbf{x}) k \mathbf{w} k x_1}}$. Next, we analyze this expectation.

If $jv_1 x_1 j > jv_2 x_2 j$, then $\text{sgn}(\mathbf{v}^\top \mathbf{x}) = \text{sgn}(v_1 x_1)$, and

$$g(x_1, x_2) + g(x_1, -x_2) = 0.$$

If $jv_2 x_2 j > jv_1 x_1 j$, then $\text{sgn}(\mathbf{v}^\top \mathbf{x}) = \text{sgn}(v_2 x_2)$, and

$$g(x_1, x_2) + g(x_1, -x_2) = \frac{jv_2 x_2 j}{1 + e^{k \mathbf{w} k x_1}} + \frac{jv_2 x_2 j}{1 + e^{-k \mathbf{w} k x_1}} = jv_2 x_2 j \quad 0.$$

Therefore, we have $g(x_1, x_2) + g(x_1, -x_2) = jv_2 x_2 j \mathbb{1}_{\{jv_2 x_2 j > jv_1 x_1 j\}}$. Further due to the spherical symmetry of the distribution D , the marginal distribution $D_{\mathbf{w}, \mathbf{v}}$ satisfies the following property: the probability density at any two points (x_1, x_2) and $(x_1, -x_2)$ are the same. Then

$$\begin{aligned} \rho &= \mathbb{E}_{\mathbf{x}} \int_{D_{\mathbf{w}, \mathbf{v}}} g(x_1, x_2) = \frac{1}{2} \mathbb{E}_{\mathbf{x}} \int_{D_{\mathbf{w}, \mathbf{v}}} jv_2 x_2 j \mathbb{1}_{\{jv_2 x_2 j > jv_1 x_1 j\}} \\ &= \frac{jv_2 j}{4\pi} \int_0^1 r \int_{\tan^{-1} \theta(\mathbf{w}, \mathbf{v})}^{\tan^{-1} \theta_j} j \sin \theta j d\theta dF(r) \\ &= \frac{\sin^2 \theta(\mathbf{w}, \mathbf{v})}{\pi} \int_0^1 r dF(r) \\ &= \frac{c_0 \sin^2 \theta(\mathbf{w}, \mathbf{v})}{\pi}, \end{aligned}$$

where $F(r)$ is the cumulative distribution function of $r := k\mathbf{x}k$ and $c_0 = \mathbb{E}_{\mathbf{x}} \int_{D_{\mathbf{w}, \mathbf{v}}} k\mathbf{x}k = \int_0^1 r dF(r)$.

Next, we prove Eq. (6). If $\theta(\mathbf{w}, \mathbf{v}) = 0$, then Eq. (6) holds from (ii) in Proposition 3. From now on, we assume $\theta(\mathbf{w}, \mathbf{v}) \neq 0$. Denote

$$\hat{\mathbf{u}} := \frac{c_0}{\pi} \mathbf{v} + r L(\mathbf{w}), \quad \hat{\mathbf{W}} := I_d - \bar{\mathbf{w}} \bar{\mathbf{w}}^\top. \quad (23)$$

Based on Eq. (5), i.e., the first result in Lemma 2, we obtain

$$\begin{aligned} \mathbf{v}^\top (I_d - \bar{\mathbf{w}} \bar{\mathbf{w}}^\top) r L(\mathbf{w}) &= \frac{c_0}{\pi} \mathbf{v}^\top (I_d - \bar{\mathbf{w}} \bar{\mathbf{w}}^\top) \mathbf{v}. \\ \Rightarrow 0 &= \mathbf{v}^\top (I_d - \bar{\mathbf{w}} \bar{\mathbf{w}}^\top) \left(\frac{c_0}{\pi} \mathbf{v} + r L(\mathbf{w}) \right) \stackrel{(23)}{=} \mathbf{v}^\top \hat{\mathbf{W}} \hat{\mathbf{u}}. \end{aligned}$$

Therefore

$$\hat{\mathbf{W}} \hat{\mathbf{u}} \perp \mathbf{v}. \quad (24)$$

We also have

$$\mathbf{w}^\top \hat{\mathbf{W}} = 0 \Rightarrow \mathbf{w}^\top \hat{\mathbf{W}} \hat{\mathbf{u}} = 0 \Rightarrow \hat{\mathbf{W}} \hat{\mathbf{u}} \perp \mathbf{w}. \quad (25)$$

Moreover, we notice that $r L(\mathbf{w}) \in \text{span}\{\bar{\mathbf{w}}, \mathbf{v}\}$, thus $\hat{\mathbf{u}} \in \text{span}\{\bar{\mathbf{w}}, \mathbf{v}\}$, which further implies

$$\hat{\mathbf{W}} \hat{\mathbf{u}} \in \text{span}\{\bar{\mathbf{w}}, \mathbf{v}\}. \quad (26)$$

Combining Eqs. (24), (25) and (26), we obtain $\hat{\mathbf{W}} \hat{\mathbf{u}} = 0$, i.e.,

$$(I_d - \bar{\mathbf{w}} \bar{\mathbf{w}}^\top) \left(\frac{c_0}{\pi} \mathbf{v} + r L(\mathbf{w}) \right) = \mathbf{0}.$$

Thus we have proved Eq. (6).

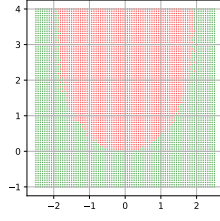


Figure 2: Verification of $\text{sgn}(g(\mathbf{w}))$ when $\mathbf{x} \sim U(S^1)$ and $\mathbf{v} = (0, 1)^\top$. The sign of $g(\mathbf{w})$ at each \mathbf{w} is estimated by 1000 random samples. “Red” means positive sign while “Green” is negative sign.

A.5 Proof of Lemma 3

Proof: Proof of (i): The first proposition follows from

$$g(\mathbf{w}) = \mathbf{w}^\top r L(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim D} \frac{y(\mathbf{x}) \mathbf{w}^\top \mathbf{x}}{1 + e^{y(\mathbf{x}) \mathbf{w}^\top \mathbf{x}}}.$$

Use the fact $x/(1 + e^x) < 0.3$, then $g(\mathbf{w}) < 0.3$.

Proof of (ii) and (iii).

$$g(\mathbf{w}) \stackrel{(20)}{=} \mathbb{E}_{\mathbf{x} \sim D_{\mathbf{w}, \mathbf{v}}} \frac{y(\mathbf{x}) \mathbf{w}^\top \mathbf{x}}{1 + e^{y(\mathbf{x}) \mathbf{w}^\top \mathbf{x}}}. \quad (27)$$

Using the spherically symmetric assumption, $D_{\mathbf{w}, \mathbf{v}}$ has the same distribution as D_2 , which is also spherically symmetric. Without loss of generality, we can assume $\mathbf{v} = (1, 0)$, $\bar{\mathbf{w}} = (w_1, w_2)$, $w_2 > 0$ (i.e., $w_i = v_i = 0, i = 3$). Then

$$g(\mathbf{w}) \stackrel{(27)}{=} \mathbb{E}_{\mathbf{x} \sim D_2} \frac{\text{sgn}(x_1) \|\mathbf{w}\| \|\bar{\mathbf{w}}\| \mathbf{x}^\top \bar{\mathbf{w}}}{1 + e^{\text{sgn}(x_1) \|\mathbf{w}\| \|\bar{\mathbf{w}}\| \mathbf{x}^\top \bar{\mathbf{w}}}}. \quad (28)$$

Note that $\bar{\mathbf{w}}^\top \mathbf{x} = w_1 x_1 + \sqrt{1 - w_1^2} x_2$, which is only related to $w_1 = \cos \theta(\mathbf{w}, \mathbf{v})$. Hence, $g(\mathbf{w})$ is a function only decided by $\|\mathbf{w}\|$ and $\theta(\mathbf{w}, \mathbf{v})$ from Eq. (28).

Now we reuse the spherically symmetric assumption, and assume $\bar{\mathbf{w}} = (1, 0)$, $\mathbf{v} = (v_1, v_2)$. Then $jv_2/v_1j = j \tan \theta(\mathbf{w}, \mathbf{v})j$, and

$$\frac{g(\mathbf{w})}{\|\mathbf{w}\|} \stackrel{(27)}{=} \mathbb{E}_{\mathbf{x} \sim D_2} h(x_1, x_2), \text{ where } h(x_1, x_2) = \frac{\text{sgn}(\mathbf{v}^\top \mathbf{x}) x_1}{1 + e^{\text{sgn}(\mathbf{v}^\top \mathbf{x}) \|\mathbf{w}\| x_1}}. \quad (29)$$

Case 1. If $jv_1 x_1j > jv_2 x_2j$, i.e., $jx_1j > jx_2 \tan \theta(\mathbf{w}, \mathbf{v})j$, then $\text{sgn}(\mathbf{v}^\top \mathbf{x}) = \text{sgn}(v_1 x_1)$,

$$h(x_1, x_2) + h(x_1, -x_2) = \frac{2 \text{sgn}(v_1) jx_1j}{1 + e^{k\mathbf{w}\| \text{sgn}(v_1) jx_1j}} > 0, \text{ if } x_1 \neq 0.$$

Case 2. If $jv_2 x_2j > jv_1 x_1j$, i.e., $jx_1j < jx_2 \tan \theta(\mathbf{w}, \mathbf{v})j$, then $\text{sgn}(\mathbf{v}^\top \mathbf{x}) = \text{sgn}(v_2 x_2)$,

$$h(x_1, x_2) + h(x_1, -x_2) = \frac{1 - e^{k\mathbf{w}\| x_1}}{1 + e^{k\mathbf{w}\| x_1}} x_1 < 0, \text{ if } x_1 \neq 0.$$

The marginal distribution $D_{\mathbf{w}, \mathbf{v}}$ satisfies the following property: the probability density at any two points (x_1, x_2) and $(x_1, -x_2)$ are the same. Combining Case 1 and Case 2, we could obtain

$$\frac{g(\mathbf{w})}{\|\mathbf{w}\|} = \underbrace{\mathbb{E}_{\mathbf{x} \sim D_2} \frac{\text{sgn}(v_1) jx_1j}{1 + e^{k\mathbf{w}\| \text{sgn}(v_1) jx_1j}} \mathbb{1}_{|jx_1j| > |jx_2 \tan \theta|}}_{J_1} + \underbrace{\mathbb{E}_{\mathbf{x} \sim D_2} \frac{1 - e^{k\mathbf{w}\| x_1}}{1 + e^{k\mathbf{w}\| x_1}} \frac{x_1}{2} \mathbb{1}_{|jx_1j| < |jx_2 \tan \theta|}}_{J_2}, \quad (30)$$

where $v_1 = \cos \theta$ with $\theta = \theta(\mathbf{w}, \mathbf{v})$.

Proof of (ii): When $\theta(\mathbf{w}, \mathbf{v}) > \pi/2$, then $v_1 < 0$, $J_1 < 0$ and $J_2 < 0$, showing that $g(\mathbf{w}) < 0$.

Proof of (iii): When $\theta(\mathbf{w}, \mathbf{v}) = \pi/2$, i.e., $v_1 = 0$. Then $J_1 = 0$ and $J_2 < 0$. Note that $g(\mathbf{w})$ is a function only decided by $\|\mathbf{w}\|$ and $\theta(\mathbf{w}, \mathbf{v})$ from the above argument. We redefine $g(\mathbf{w}) = g(\|\mathbf{w}\|, \theta)$ with $\theta = \theta(\mathbf{w}, \mathbf{v})$. Denoting $u(\mathbf{x}) := \frac{jx_1j}{1+e^{\|\mathbf{w}\|jx_1j}} - \frac{1}{1+e^{\|\mathbf{w}\|kx_1}} \frac{x_1}{2}$, $u(\mathbf{x}) > 0$ if $x_1 \neq 0$. Then for $\theta = \pi/2$, and $\theta > \delta > 0$, and some $R > r > 0$,

$$\begin{aligned} \frac{g(\|\mathbf{w}\|, \theta - \delta) - g(\|\mathbf{w}\|, \theta)}{\|\mathbf{w}\|} &= \mathbb{E}_{\mathbf{x}} D_2 u(\mathbf{x}) \mathbb{1}_{f_{jx_2} \tan \theta_j - jx_1j > jx_2 \tan(\theta - \delta)j} \\ &= 2 \mathbb{E}_{\mathbf{x}} D_2 u(\mathbf{x}) \mathbb{1}_{f_{x_2/x_1} = \tan \alpha, \pi/2 - \theta < \alpha < \pi/2 - (\theta - \delta)} \\ &= 2 \mathbb{E}_{\mathbf{x}} D_2 u(\mathbf{x}) \mathbb{1}_{f_{x_2/x_1} = \tan \alpha, \pi/2 - \theta < \alpha < \pi/2 - (\theta - \delta)} \mathbb{1}_{r^2 - x_1^2 + x_2^2 > R^2} \\ &= 2 \mathbb{E}_{\mathbf{x}} D_2 \epsilon \mathbb{1}_{f_{x_2/x_1} = \tan \alpha, \pi/2 - \theta < \alpha < \pi/2 - (\theta - \delta)} = \frac{2\delta\epsilon}{\pi}, \end{aligned} \quad (31)$$

where we adopt

$$\epsilon = \min_{\substack{r^2 - x_1^2 + x_2^2 > R^2 \\ x_2/x_1 = \tan \alpha \\ \pi/2 - \theta < \alpha < \pi/2 - (\theta - \delta)}} \frac{jx_1j}{1 + e^{\|\mathbf{w}\|jx_1j}} - \frac{1}{1 + e^{\|\mathbf{w}\|kx_1}} \frac{x_1}{2} > 0,$$

since $\pi/2 - (\theta - \delta) < \pi/2$, and ϵ is related to $R, r, \|\mathbf{w}\|$. Hence, we obtain

$$\frac{\partial g(\|\mathbf{w}\|, \theta)}{\partial \theta} = \lim_{\delta \downarrow 0} \frac{g(\|\mathbf{w}\|, \theta) - g(\|\mathbf{w}\|, \theta - \delta)}{\delta} = \frac{2\epsilon}{\pi} \|\mathbf{w}\| < 0.$$

We conclude

$$\frac{\partial g(\mathbf{w})}{\partial \theta(\mathbf{w}, \mathbf{v})} < 0.$$

Proof of (iv): We fix $\theta(\mathbf{w}, \mathbf{v})$ or $\bar{\mathbf{w}}$, and consider

$$\bar{g}(r) := \frac{g(\mathbf{w})}{\|\mathbf{w}\|} = \mathbb{E}_{\mathbf{x}} D_2 \frac{y(\mathbf{x}) \bar{\mathbf{w}}^{\top} \mathbf{x}}{1 + e^{y(\mathbf{x}) r \bar{\mathbf{w}}^{\top} \mathbf{x}}}.$$

Then when $r \neq 0$, we obtain

$$\bar{g}(0) = \lim_{r \downarrow 0} \bar{g}(r) = \frac{1}{2} \mathbb{E}_{\mathbf{x}} D_2 y(\mathbf{x}) \bar{\mathbf{w}}^{\top} \mathbf{x} = \frac{c_0 \cos \theta(\mathbf{w}, \mathbf{v})}{\pi}.$$

In addition,

$$\left| \frac{\partial \bar{g}(r)}{\partial r} \right| = \left| \mathbb{E}_{\mathbf{x}} D_2 \frac{(\bar{\mathbf{w}}^{\top} \mathbf{x})^2 e^{y(\mathbf{x}) r \bar{\mathbf{w}}^{\top} \mathbf{x}}}{(1 + e^{y(\mathbf{x}) r \bar{\mathbf{w}}^{\top} \mathbf{x}})^2} \right| \leq \frac{1}{4} \mathbb{E}_{\mathbf{x}} D_2 (\bar{\mathbf{w}}^{\top} \mathbf{x})^2 = \frac{1}{8} \mathbb{E}_{\mathbf{x}} D_2 \|\mathbf{x}\|^2.$$

Then $\bar{g}(r)$ is L -Lipschitz continuous with $L := \frac{1}{8} \mathbb{E}_{\mathbf{x}} D_2 \|\mathbf{x}\|^2$, hence when $r = \frac{c_0 \cos \theta(\mathbf{w}, \mathbf{v}) j}{2\pi L}$, we obtain

$$j\bar{g}(r) - \bar{g}(0)j \leq rL \left(\frac{c_0 \cos \theta(\mathbf{w}, \mathbf{v}) j}{2\pi} \right) \Rightarrow \bar{g}(r) - \bar{g}(0) > 0 \Rightarrow g(\mathbf{w}) \cos \theta(\mathbf{w}, \mathbf{v}) > 0.$$

A.6 Proof of Lemma 1

Lemma 9 (Restatement of Lemma 1) Denote $g(\mathbf{w}) = \frac{1}{2} \frac{\partial k\mathbf{w}(t)k^2}{\partial t}$, and denote $T = \inf t : \frac{\partial k\mathbf{w}(t)k^2}{\partial t} = 0$. Under Assumption 1, we have $T < 1$, and

$$g(\mathbf{w}(t)) = \frac{1}{2} \frac{\partial k\mathbf{w}(t)k^2}{\partial t} \begin{cases} 0, & \delta t = T \\ < 0, & \delta t < T. \end{cases} \quad (32)$$

Proof of Lemma 1: The proof will be divided into three steps.

Step 1: We first present an auxiliary lemma, whose proof is given in Appendix A.6.1.

Lemma 10 Consider a differentiable function $h(t), t \geq 0$. Suppose for any t_0 such that $h(t_0) = 0$, we must have $h'(t_0) > 0$; in other words,

$$h(t_0) = 0 \implies h'(t_0) > 0. \quad (33)$$

Denote $T = \inf \{t : h(t) = 0\}$. Then

$$h(t) \begin{cases} \geq 0, & \forall t \geq T \\ < 0, & \forall t < T. \end{cases} \quad (34)$$

Step 2: Verify that $h(t) := g(\mathbf{w}(t))$ satisfies the condition of the auxiliary lemma.

More precisely, we will show $h(t) := g(\mathbf{w}(t))$ satisfies Eq. (33), i.e., $h(t_0) = 0$ implies $h'(t_0) > 0$.

Suppose $h(t_0) = 0$. According to Lemma 3 (ii) that $g(\mathbf{w}) < 0$ if $\theta(\mathbf{w}, \mathbf{v}) > \pi/2$, we obtain that $g(\mathbf{w}(t_0)) = 0$ implies

$$\theta(\mathbf{w}(t_0), \mathbf{v}) = \pi/2. \quad (35)$$

Recall that Lemma 3 (iii) states that $\theta(\mathbf{w}, \mathbf{v}) = \pi/2$ implies $\partial g(\mathbf{w})/\partial \theta(\mathbf{w}, \mathbf{v}) < 0$. By Eq. (35)

$$\left. \frac{\partial g(\mathbf{w})}{\partial \theta} \right|_{\mathbf{w}=\mathbf{w}(t_0)} < 0. \quad (36)$$

From Lemma 2 and Eq. (4), we have

$$\frac{\partial \cos \theta(t)}{\partial t} = \frac{c_0}{\pi k \mathbf{w}(t) k} \sin^2 \theta(t), \forall t \geq 0.$$

As mentioned in Remark 2, we can assume the $k \mathbf{w}(t) k \neq 0, \forall t \geq 0$. Then by Lemma 11, we obtain $\partial \cos \theta(t)/\partial t > 0$, thus

$$\frac{\partial \theta(t)}{\partial t} < 0, \forall t \geq 0. \quad (37)$$

As discussed in the Proof of Lemma 3, $g(\mathbf{w}(t))$ can be viewed as a function only decided by $k \mathbf{w}(t) k$ and $\theta(t)$. Then by the chain rule we have

$$\begin{aligned} h'(t_0) &= \frac{\partial g(\mathbf{w}(t_0))}{\partial t} = \frac{\partial g(\mathbf{w}(t_0))}{\partial k \mathbf{w} k} \frac{\partial k \mathbf{w}(t_0) k}{\partial t} + \frac{\partial g(\mathbf{w}(t_0))}{\partial \theta} \frac{\partial \theta(t_0)}{\partial t} \\ &= \frac{\partial g(\mathbf{w}(t_0))}{\partial \theta} \frac{\partial \theta(t_0)}{\partial t} > 0. \end{aligned}$$

Here the second equality follows from $\frac{\partial k \mathbf{w}(t_0) k}{\partial t} = 2g(\mathbf{w}(t_0)) = 0$ implies $k \mathbf{w}(t_0) k \frac{\partial k \mathbf{w}(t_0) k}{\partial t} = 0$ and the assumption that $k \mathbf{w}(t) k \neq 0, \forall t \geq 0$ by Remark 2; the final inequality is due to Eqs. (36) and (37). Therefore we have proved the condition Eq. (33).

Then by Lemma 10, we immediately obtain Eq. (32).

Step 3: prove $T < 1$. We prove by contradiction.

Assume the contrary, that $T = 1$, then $\partial k \mathbf{w}(t) k / \partial t = 0, \forall t \geq 0$. This implies $k \mathbf{w}(t) k = k \mathbf{w}(0) k, \forall t \geq 0$. However, $k \mathbf{w}(t) k$ is unbounded by (iii) in Proposition 2, a contradiction. Thus the assumption does not hold, and we must have $T < 1$. This finishes the proof.

A.6.1 Proof of Auxiliary Lemmas

Proof idea of Lemma 10: The condition Eq. (33) says that whenever the function value reaches 0, it has to ‘‘bounce up’’ (strictly increasing at that point). As a result, once the function value goes above 0, it will never drop down below 0.

Proof of Lemma 10: If $T = 1$, i.e., $h(t) < 0, \forall t \geq 0$, then Eq. (34) automatically holds. From now on, we assume $T < 1$. By the continuity of h , we have $h(T) = 0$; then by Eq. (33), we have $h'(T) > 0$.

We prove Eq. (34) by contradiction. Assume the contrary, that there exists $t_0 > T$, such that $h(t_0) < 0$. Since $h'(T) > 0$, there exists $\epsilon < t_0 - T$ such that $h(T + \epsilon) > 0$. In the interval

$[T + \epsilon, t_0]$, the left end has positive value and the right end has negative value. Thus there exists at least one root. Denote the set of all roots in this interval as K , and denote $t_m := \sup\{t : t \in K\}$. By the continuity of h , we must have $t_m \in K$ (otherwise, assume $t_m \notin K$, then there is a sequence of roots converging to t_m , implying t_m is also a root and thus $t_m \in K$, a contradiction). Then $h(t_m) = 0$ and, by Eq. (33), $h'(t_m) > 0$. By the same argument as before, there exists a certain $\epsilon_m < t_0 - t_m$ such that $h(t_m + \epsilon_m) > 0$, which further implies there is a root t_p in the interval $[t_m + \epsilon_m, t_0]$. This is a contradiction to the definition that t_m is the maximum root on $[T + \epsilon, t_0]$. This contradiction means that there does not exist any $t_0 > T$ such that $h(t_0) < 0$; in other words, $h(t) \geq 0, \forall t > T$.

Lemma 11 Suppose $f : [0, 1) \rightarrow (0, 1)$ be a continuous function. Consider the initial value problem

$$\theta(0) = \theta_0, \frac{\partial \cos \theta(t)}{\partial t} = f(t) \sin^2 \theta(t), \forall t \geq 0. \quad (38)$$

If $\theta_0 \notin 0, \pi$, then $\theta(t) \notin 0, \pi$ and $\frac{\partial \cos \theta(t)}{\partial t} > 0, \forall t \geq 0$.

Proof: The solution to Eq. (38) is

$$\frac{1}{2} \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} - \frac{1}{2} \ln \frac{1 + \cos \theta_0}{1 - \cos \theta_0} = \int_0^t f(s) ds < 1.$$

Since $\theta_0 \notin 0, \pi$, then $-1 < \ln \frac{1 + \cos \theta_0}{1 - \cos \theta_0} < 1$. Thus $-1 < \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} < 1$, leading to $\theta(t) \notin 0, \pi$. Hence $\sin \theta(t) \neq 0$, showing that $\frac{\partial \cos \theta(t)}{\partial t} > 0$.

A.7 Proofs of Theorem 1

Proof: When $t < T$, by Lemma 2, we have

$$\frac{\partial}{\partial t} \cos(\theta(t)) = \frac{1}{k\mathbf{w}(t)k} \frac{c_0 \sin^2 \theta(t)}{\pi} = \frac{1}{k\mathbf{w}(0)k} \frac{c_0 \sin^2 \theta(t)}{\pi}.$$

We obtain

$$\frac{1}{2} \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} - \frac{1}{2} \ln \frac{1 + \cos \theta(t_0)}{1 - \cos \theta(t_0)} = \frac{c_0}{k\mathbf{w}(0)k\pi} t.$$

Thus

$$\cos \theta(t) = 1 - \frac{2}{e^{A_1 t + B_1} + 1}, t < T.$$

When $t > T$, based on (i) in Lemma 3,

$$\frac{\partial}{\partial t} k\mathbf{w}(t)k^2 = 2g(\mathbf{w}(t)) \approx 0.6.$$

This implies $k\mathbf{w}(t)k^2 \approx 0.6(t - T) + k\mathbf{w}(T)k^2$. Together with Lemma 2, we have

$$\frac{\partial}{\partial t} \cos(\theta(t)) = \frac{1}{k\mathbf{w}(t)k} \frac{c_0 \sin^2 \theta(t)}{\pi} = \frac{1}{\sqrt{0.6(t - T) + k\mathbf{w}(T)k^2}} \frac{c_0 \sin^2 \theta(t)}{\pi}.$$

After taking the integral, we have

$$\frac{1}{2} \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} - \frac{1}{2} \ln \frac{1 + \cos \theta(T)}{1 - \cos \theta(T)} = \frac{2c_0}{0.6\pi} \left(\sqrt{0.6(t - T) + k\mathbf{w}(T)k^2} - k\mathbf{w}(T)k \right).$$

Thus

$$\cos \theta(t) = 1 - \frac{2}{e^{A_2 \sqrt{t - T + C_2} + B_2} + 1}, t > T.$$

Remark 1: Based on Lemma 1, $k\mathbf{w}(t)k$ first decreases when $t > T$, then increases when $t < T$. Thus the approach for analyzing the phase $t > T$ above cannot be applied to the phase $t < T$.

Remark 2: The convergence rate analysis for the stage $t > T$ can be applied to any $t \geq 0$. Anyhow, we choose a different approach for the stage $t < T$ since it leads to a stronger bound of the convergence rate.

A.8 Proof of Lemma 4

Lemma 12

$$k\mathbf{w}(n+1)k^2 = (\bar{\mathbf{w}}(n) \succ \mathbf{w}(n+1))^2 + \left(\frac{c_0\eta_n}{\pi}\right)^2 \sin^2 \theta(n).$$

Proof:

$$\begin{aligned} k\mathbf{w}(n+1)k^2 &= k\mathbf{w}(n)k^2 - 2\eta_n \mathbf{w}(n) \succ r L(\mathbf{w}(n)) + \eta_n^2 k r L(\mathbf{w}(n))k^2 \\ &= (k\mathbf{w}(n)k - \eta_n \bar{\mathbf{w}}(n) \succ r L(\mathbf{w}(n)))^2 + \eta_n^2 \|(I - \bar{\mathbf{w}}(n) \bar{\mathbf{w}}(n) \succ) r L(\mathbf{w}(n))\|^2 \\ &\stackrel{(1)}{=} (k\mathbf{w}(n)k - \eta_n \bar{\mathbf{w}}(n) \succ r L(\mathbf{w}(n)))^2 + \left(\frac{c_0\eta_n}{\pi}\right)^2 \|(I - \bar{\mathbf{w}}(n) \bar{\mathbf{w}}(n) \succ) \mathbf{v}\|^2 \\ &= (k\mathbf{w}(n)k - \eta_n \bar{\mathbf{w}}(n) \succ r L(\mathbf{w}(n)))^2 + \left(\frac{c_0\eta_n}{\pi}\right)^2 \sin^2 \theta(n) \\ &= (\bar{\mathbf{w}}(n) \succ \mathbf{w}(n+1))^2 + \left(\frac{c_0\eta_n}{\pi}\right)^2 \sin^2 \theta(n). \end{aligned}$$

The equality in (1) follows from Lemma 2.

Now we turn to the main proof of Lemma 4.

Proof: From (ii) in Lemma 3, when $\cos \theta(n) = 0$, $\mathbf{w}(n) \succ r L(\mathbf{w}(n)) = 0$, leading to

$$k\mathbf{w}(n+1)k^2 = k\mathbf{w}(n)k^2 - 2\eta_n \mathbf{w}(n) \succ r L(\mathbf{w}(n)) + \eta_n^2 k r L(\mathbf{w}(n))k^2 = k\mathbf{w}(n)k^2 + \eta_n^2 c_0^2.$$

From Lemma 12, $k\mathbf{w}(n+1)k = |\bar{\mathbf{w}}(n) \succ \mathbf{w}(n+1)| = k\mathbf{w}(n)k - \eta_n \bar{\mathbf{w}}(n) \succ r L(\mathbf{w}(n))$, then

$$\begin{aligned} \cos \theta(n+1) - \cos \theta(n) &= \frac{1}{k\mathbf{w}(n+1)k} (\mathbf{v} \succ (\mathbf{w}(n) - \eta_n r L(\mathbf{w}(n))) - k\mathbf{w}(n+1)k \mathbf{v} \succ \bar{\mathbf{w}}(n)) \\ &= \frac{1}{k\mathbf{w}(n+1)k} (\mathbf{v} \succ (\mathbf{w}(n) - \eta_n r L(\mathbf{w}(n))) - (k\mathbf{w}(n)k - \eta_n \bar{\mathbf{w}}(n) \succ r L(\mathbf{w}(n))) \mathbf{v} \succ \bar{\mathbf{w}}(n)) \\ &= \frac{1}{k\mathbf{w}(n+1)k} \left(\eta_n (\mathbf{v} - (\bar{\mathbf{w}}(n) \succ \mathbf{v}) \bar{\mathbf{w}}(n)) \succ r L(\mathbf{w}(n)) \right) \\ &= \frac{1}{k\mathbf{w}(n+1)k} \frac{c_0\eta_n \sin^2 \theta(n)}{\pi} \frac{\eta_n}{\pi \sqrt{A + \sum_{i=0}^n \eta_i^2}} \sin^2 \theta(n) > 0. \end{aligned}$$

Thus $\theta(n)$ decrease when n increase. i.e. $\frac{\pi}{2} - \theta(n+1) < \theta(n)$ and $0 < \cos \theta(n+1) < \cos \theta(n)$. Then we obtain

$$\cos \theta(n+1) - \cos \theta(n) = \frac{\eta_n (1 - \cos \theta(n))(1 + \cos \theta(n))}{\pi \sqrt{A + \sum_{i=0}^n \eta_i^2}} (1 - \cos \theta(n)) \frac{B\eta_n}{\sqrt{A + \sum_{i=0}^n \eta_i^2}}.$$

Adjust the terms, we get

$$\left(1 - \frac{B\eta_n}{\sqrt{A + \sum_{i=0}^n \eta_i^2}}\right) (1 - \cos \theta(n)) < 1 - \cos \theta(n+1),$$

showing that

$$\ln(1 - \cos \theta(n)) < \ln(1 - \cos \theta(n+1)) \quad \ln \left(1 - \frac{B\eta_n}{\sqrt{A + \sum_{i=0}^n \eta_i^2}}\right) < \frac{B\eta_n}{\sqrt{A + \sum_{i=0}^n \eta_i^2}}.$$

Hence,

$$\cos \theta(n) < 1 - (1 - \cos \theta(0)) e^{-BS_n}.$$

When $S_n \rightarrow \infty$, for some finite T steps, $\cos \theta(T) = 0$, giving that $\mathbf{v} \succ \mathbf{w}(T) = 0$.

Remark 6 Obviously, $S_n < n$, and we list several choices of $\{\eta_n\}_{i=1}^n$:

$$S_n = \begin{cases} (n), & \eta_n = (q^n), q > 1; \\ (n^{\min\{\alpha+1, 1/2\}}), & \eta_n = (n^\alpha), 1 < \alpha, \alpha \neq 1/2; \\ (\ln(n)), & \eta_n = (n^{-1}); \\ < 1, & \eta_n = (n^\alpha), \alpha < -1. \end{cases}$$

A.9 Proof of Lemma 5

Proof: Use the upper bound of $k\mathbf{w}(n+1)k$ below

$$k\mathbf{w}(n+1)k^2 = k\mathbf{w}(n)k^2 - 2\eta_n \mathbf{w}(n)^\top \nabla L(\mathbf{w}_n) + \eta_n^2 k \nabla L(\mathbf{w}_n)^\top \nabla L(\mathbf{w}_n) k^2 \leq k\mathbf{w}(n)k^2 + 0.6\eta_n + c_0^2 \eta_n^2.$$

Therefore, from Lemma 12, we get

$$\begin{aligned} \cos \theta(n+1) - \cos \theta(n) &= \frac{\mathbf{v}^\top \mathbf{w}(n+1) - k\mathbf{w}(n+1)k \mathbf{v}^\top \bar{\mathbf{w}}(n)}{k\mathbf{w}(n+1)k} \\ &= \frac{(\mathbf{v}^\top \bar{\mathbf{w}}(n) \bar{\mathbf{w}}(n))^\top \mathbf{w}(n+1) - (k\mathbf{w}(n+1)k - \bar{\mathbf{w}}(n)^\top \mathbf{w}(n+1)) \mathbf{v}^\top \bar{\mathbf{w}}(n)}{k\mathbf{w}(n+1)k} \\ &= \frac{1}{k\mathbf{w}(n+1)k} \left(\eta_n (\mathbf{v}^\top \bar{\mathbf{w}}(n) \bar{\mathbf{w}}(n))^\top \nabla L(\mathbf{w}(n)) \right. \\ &\quad \left. - \frac{k\mathbf{w}(n+1)k^2 - (\bar{\mathbf{w}}(n)^\top \mathbf{w}(n+1))^2}{k\mathbf{w}(n+1)k + \bar{\mathbf{w}}(n)^\top \mathbf{w}(n+1)} \cos \theta(n) \right) \\ &= \frac{1}{k\mathbf{w}(n+1)k} \left(\frac{c_0 \eta_n \sin^2 \theta(n)}{\pi} - \frac{c_0^2 \eta_n^2 \sin^2 \theta(n) \cos \theta(n) / \pi^2}{k\mathbf{w}(n+1)k + \bar{\mathbf{w}}(n)^\top \mathbf{w}(n+1)} \right) \\ &\stackrel{(i)}{=} \frac{1}{k\mathbf{w}(n+1)k} \frac{\delta c_0 \eta_n \sin^2 \theta(n)}{(1+\delta)\pi} - \frac{\delta \eta_n}{(1+\delta)\pi \sqrt{A + \sum_{i=0}^n \eta_i^2 + C\eta_i}} \sin^2 \theta(n), \end{aligned}$$

where (i) uses the sufficient convergence condition in Eq. (11). Hence, similarly, we obtain

$$\cos \theta(n) - 1 \leq (1 - \cos \theta(0)) e^{-BS_n^+}.$$

Remark 7 Obviously, $S_n^+ < n$, and we list several choices of $\{\eta_n\}_{i=1}^n$:

$$S_n^+ = \begin{cases} (n), & \eta_n = (q^n), q > 1; \\ (n^{(\min\{\alpha, 0.9\})/2}), & \eta_n = (n^\alpha), 1 < \alpha; \\ (\sqrt{\ln(n)}), & \eta_n = (n^{-1}); \\ < 1, & \eta_n = (n^\alpha), \alpha < -1. \end{cases}$$

A.10 Proof of Theorem 2

Proof: Note that $k\mathbf{w}(n)k - k\mathbf{w}(n)k \cos \theta(n) = \mathbf{v}^\top \mathbf{w}(n)$. From (ii) in Proposition 2 and (iii) in Proposition 3, the right term monotonically increases to infinity. Thus, after finite iterations, the sufficient convergence condition would be satisfied: $k\mathbf{w}(n_1)k \cos \theta(n_1) \geq R_1 = \eta_+ c_0 + c_0 \eta_+ / \pi$. Therefore, we have $k\mathbf{w}(n)k - k\mathbf{w}(n)k \cos \theta(n) \geq k\mathbf{w}(n_1)k \cos \theta(n_1) \geq R_1$, $\delta n > n_1$. Thus, from $\eta_+ \eta_n$, we obtain $\delta n > n_1$,

$$k\mathbf{w}(n+1)k + \bar{\mathbf{w}}(n)^\top \mathbf{w}(n+1) \leq 2(k\mathbf{w}(n)k - \eta_n k \nabla L(\mathbf{w}(n))k) + 2c_0 \eta_n \cos \theta(n) / \pi,$$

which satisfies the sufficient convergence condition with $\delta = 1$. Hence, from Lemma 5, we obtain superpolynomial directional convergence of $1 - \cos \theta(n)$ from n_1 .

B Missing Proofs in Section 4

B.1 Proof of Lemma 7

Proof: From the induced flow on \mathbf{w}_e , we obtain

$$\begin{aligned} &(\mathbf{v}^\top \bar{\mathbf{w}}_e \bar{\mathbf{w}}_e)^\top \frac{\partial \mathbf{w}_e(t)}{\partial t} \\ &\stackrel{(14)}{=} (\mathbf{v}^\top \bar{\mathbf{w}}_e \bar{\mathbf{w}}_e)^\top k\mathbf{w}_e k^2 \frac{2}{9} \left(\frac{dL^1(\mathbf{w}_e)}{d\mathbf{w}} + (N-1) \bar{\mathbf{w}}_e \bar{\mathbf{w}}_e^\top \frac{dL^1(\mathbf{w}_e)}{d\mathbf{w}} \right) \\ &= k\mathbf{w}_e k^2 \frac{2}{9} (\mathbf{v}^\top \bar{\mathbf{w}}_e \bar{\mathbf{w}}_e)^\top \frac{dL^1(\mathbf{w}_e)}{d\mathbf{w}} = k\mathbf{w}_e k^2 \frac{2}{9} \frac{c_0 \sin^2 \theta(\mathbf{w}_e, \mathbf{v})}{\pi}. \end{aligned}$$

Therefore we have

$$\frac{\partial \cos \theta(\mathbf{w}_e(t), \mathbf{v})}{\partial t} = \frac{(\mathbf{v} \cdot \overline{\mathbf{w}}_e(t) - \|\mathbf{v}\| \|\overline{\mathbf{w}}_e(t)\|)}{\|\mathbf{w}_e(t)\|^2} \frac{\partial \mathbf{w}_e(t)}{\partial t} = \frac{c_0 \sin^2 \theta(\mathbf{w}_e(t), \mathbf{v})}{\pi} \|\mathbf{w}_e(t)\|^{-\frac{2}{N}}.$$

B.2 Proof of Lemma 8

Proof: First, we establish the dynamics of the $(2/N)$ -th power of the induced norm. More specifically, we will show that if $\mathbf{w}_e(t) \notin \mathbf{0}$, then

$$\frac{\partial \|\mathbf{w}_e(t)\|^{\frac{2}{N}}}{\partial t} = \mathbb{E}_{\mathbf{x}} \left[D_2 \frac{2y(\mathbf{x})\mathbf{w}_e(t) \cdot \mathbf{x}}{1 + e^{y(\mathbf{x})\mathbf{w}_e(t) \cdot \mathbf{x}}} \right]. \quad (39)$$

This relation is proved as follows.

$$\begin{aligned} \frac{\partial \|\mathbf{w}_e(t)\|^{\frac{2}{N}}}{\partial t} &= \frac{2}{N} \|\mathbf{w}_e(t)\|^{\frac{2}{N}-1} \overline{\mathbf{w}}_e(t) \cdot \frac{\partial \mathbf{w}_e(t)}{\partial t} \\ &\stackrel{(14)}{=} \frac{2}{N} \mathbf{w}_e(t) \cdot \left(\frac{dL^1(\mathbf{w}_e(t))}{d\mathbf{w}} + (N-1) \overline{\mathbf{w}}_e(t) \overline{\mathbf{w}}_e(t) \cdot \frac{dL^1(\mathbf{w}_e(t))}{d\mathbf{w}} \right) \\ &= 2 \mathbf{w}_e(t) \cdot \frac{dL^1(\mathbf{w}_e(t))}{d\mathbf{w}} = \mathbb{E}_{\mathbf{x}} \left[D_2 \frac{2y(\mathbf{x})\mathbf{w}_e(t) \cdot \mathbf{x}}{1 + e^{y(\mathbf{x})\mathbf{w}_e(t) \cdot \mathbf{x}}} \right]. \end{aligned}$$

We now turn to prove each part of Lemma 8.

Proof of (i): From Eq. (39), we have

$$\frac{\partial \|\mathbf{w}_e(t)\|^{\frac{2}{N}}}{\partial t} = \mathbb{E}_{\mathbf{x}} \left[D_2 \frac{2y(\mathbf{x})\mathbf{w}_e(t) \cdot \mathbf{x}}{1 + e^{y(\mathbf{x})\mathbf{w}_e(t) \cdot \mathbf{x}}} \right] \geq 2 \mathbb{E}_{\mathbf{x}} \left[D_2 \|\mathbf{x}\| \|\mathbf{w}_e(t)\| \right] = 2c_0 \|\mathbf{w}_e(t)\|.$$

This implies

$$\frac{2}{N} \frac{\partial \|\mathbf{w}_e(t)\|^{\frac{2}{N}-1}}{\partial t} \geq 2c_0.$$

Integrate both sides for the range $[0, t]$, we obtain the lower bound:

$$\|\mathbf{w}_e(t)\|^{\frac{2}{N}-1} \geq \left(\|\mathbf{w}_e(0)\|^{\frac{2}{N}-1} + (N-2)c_0 t \right)^{-1}.$$

The upper bound is due to (i) in Lemma 3. More specifically,

$$\frac{\partial \|\mathbf{w}_e(t)\|^{\frac{2}{N}}}{\partial t} = \mathbb{E}_{\mathbf{x}} \left[D_2 \frac{2y\mathbf{w}_e(t) \cdot \mathbf{x}}{1 + e^{y\mathbf{w}_e(t) \cdot \mathbf{x}}} \right] \leq 0.6 \|\mathbf{w}_e(t)\| \left(\|\mathbf{w}_e(0)\|^{\frac{2}{N}} + 0.6t \right)^{\frac{N}{2}}.$$

Proof of (ii): From the lower bound in Eq. (17), we could see

$$\|\mathbf{w}_e(t)\| > 0, \quad \forall t \geq 0. \quad (40)$$

Note that

$$\begin{aligned} \frac{\partial \|\mathbf{w}_e(t)\|^2}{\partial t} &= \mathbf{w}_e(t) \cdot \frac{\partial \mathbf{w}_e(t)}{\partial t} \stackrel{(14)}{=} N \|\mathbf{w}_e(t)\|^{\frac{2}{N}} \mathbf{w}_e(t) \cdot \frac{dL^1(\mathbf{w}_e(t))}{d\mathbf{w}} \\ &\stackrel{(7)}{=} N \|\mathbf{w}_e(t)\|^{\frac{2}{N}} g(\mathbf{w}_e(t)). \end{aligned} \quad (41)$$

Here $g(\mathbf{w}) = \mathbf{w} \cdot \frac{dL^1(\mathbf{w})}{d\mathbf{w}}$ is the same as the one defined in the analysis of logistic regression. This definition applies to any \mathbf{w} , including the points on the trajectory of linear model training as well as deep linear net training. Notice that Lemma 3 is established for $g(\mathbf{w})$ with any \mathbf{w} , thus can be used for analyzing both linear model training and deep linear net training.

The rest is quite similar to Step 2 in the proof of Lemma 1 in Appendix A.6. There is a slight difference that we need to show the extra term $N \|\mathbf{w}_e(t)\|^{\frac{2}{N}}$ in Eq. (41) compared to $\frac{1}{2} \frac{\partial \|\mathbf{w}_e(t)\|^2}{\partial t} = g(\mathbf{w}_e(t))$ does not affect the argument.

From the assumption $\frac{\partial k\mathbf{w}_e(t_0)k^2}{\partial t} = 0$ and Eq. (41), we obtain $g(\mathbf{w}_e(t_0)) = 0$. Further, by (ii) in Lemma 3 we obtain $\theta(t_0) \neq \pi/2$. Note that $\partial \cos \theta(t)/\partial t > 0$ by Eq. (40), Lemmas 7 and 11, leading to $\partial \theta(t)/\partial t < 0, \forall t \geq 0$.

Moreover, we have $\mathbf{w}_e(t) \in \text{span}\{\mathbf{w}_e(0), \mathbf{v}\}$ since $rL(\mathbf{w}_e(t)) \in \text{span}\{\mathbf{w}_e(t), \mathbf{v}\}$ by Eqs. (20) and (14). Hence we can view $g(\mathbf{w}_e(t))$ as a function only decided by $k\mathbf{w}_e(t)k$ and $\theta(t)$. Using (iii) in Lemma 3, we have

$$\frac{\partial g(\mathbf{w}_e(t_0))}{\partial t} = \frac{\partial g(\mathbf{w}_e(t_0))}{\partial k\mathbf{w}_e k} \frac{\partial k\mathbf{w}_e(t_0)k}{\partial t} + \frac{\partial g(\mathbf{w}_e(t_0))}{\partial \theta} \frac{\partial \theta(t_0)}{\partial t} = \frac{\partial g(\mathbf{w}_e(t_0))}{\partial \theta} \frac{\partial \theta(t_0)}{\partial t} > 0,$$

where we use $\partial k\mathbf{w}_e(t_0)k/\partial \theta(t) = 0$ by $g(\mathbf{w}_e(t_0)) = 0$ and Eq. (40). This finishes the proof of the desired property: $\frac{\partial k\mathbf{w}_e(t_0)k^2}{\partial t} = 0$ implies $\frac{\partial g(\mathbf{w}_e(t_0))}{\partial t} > 0$.

B.3 Proof of Proposition 4

From Eq. (16) and $N = 2$, we get $\frac{\partial \cos \theta(t)}{\partial t} = \frac{c_0 \sin^2 \theta(t)}{\pi}$. Integrating from 0 to t , we obtain

$$\ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} - \ln \frac{1 + \cos \theta(0)}{1 - \cos \theta(0)} = \frac{2c_0}{\pi} t.$$

Hence

$$\cos \theta(t) = 1 - \frac{2}{C_1 e^{2c_0 t/\pi} + 1}, \text{ with } C_1 = \frac{1 + \cos \theta(0)}{1 - \cos \theta(0)}.$$

B.4 Proof of Lemma 6

Lemma 6 consists of two properties: Eq. (15) and $T < 1$.

First, Eq. (15) is an immediate consequence of Lemma 10 (given in Appendix A.6) and Part (ii) of Lemma 8. More specifically, Part (ii) of Lemma 8 essentially verifies the condition Eq. (33) in Lemma 10 for $h(t) = g(\mathbf{w}_e(t))$. Thus the conclusion of Lemma 10 holds, i.e., Eq. (15) holds.

Next, we prove $T < 1$ by contradiction. Assume the contrary, that $T = 1$, i.e.,

$$\partial k\mathbf{w}_e(t)k^2/\partial t < 0, \quad \forall t \geq 0. \quad (42)$$

Then $k\mathbf{w}_e(t)k$ converges to a finite non-negative value. There are two possibilities.

Case 1: $k\mathbf{w}_e(t)k \neq 0$. We will derive a contradiction. The intuition is that the origin is a saddle point, thus the gradient flow will not converge to it. A formal argument is the following.

We first claim that $\cos \theta(t) \neq 1$ ³. Using Lemma 7 and the lower bound of $k\mathbf{w}_e(t)k$ in Lemma 8, we obtain

$$\frac{1}{2} \ln \frac{1 + \cos \theta(t)}{1 - \cos \theta(t)} - \frac{1}{2} \ln \frac{1 + \cos \theta(0)}{1 - \cos \theta(0)} - \frac{1}{(N-2)\pi} \ln \frac{A_1 t + B_1}{B_1}.$$

This implies

$$\cos \theta(t) = 1 - \frac{2}{C_1 (A_1 t/B_1 + 1)^\alpha + 1}, \quad \forall t \geq 0. \quad (43)$$

As a consequence, there exists $t_0 > 0$ such that $\cos \theta(t) \leq \delta < 1, \forall t \geq t_0$. Since $\mathbf{w}_e(t)$ converges to origin, by (iv) in Lemma 3, for some $t_1 > t_0$ with small enough $k\mathbf{w}_e(t_1)k$, we have $g(\mathbf{w}_e(t_1)) < 0$, i.e., $\partial k\mathbf{w}_e(t_1)k^2/\partial t > 0$. This is a contradiction to Eq. (42).

Case 2: $k\mathbf{w}_e(t)k \leq \epsilon > 0$. According to (43), we have $\cos \theta(t) \neq 1$, then from (iii) of Lemma 3, we obtain

$$\liminf_{t \uparrow 1} \frac{\partial k\mathbf{w}_e(t)k^2}{\partial t} \stackrel{(41)}{=} \liminf_{t \uparrow 1} N k\mathbf{w}_e(t)k^2 \frac{2}{N} g(\mathbf{w}_e(t)) \stackrel{(30)}{=} N \epsilon^2 \frac{2}{N} \mathbb{E}_{\mathbf{x} \sim D_2} \frac{\epsilon^j x_{1j}}{1 + e^{\epsilon^j x_{1j}}} > 0. \quad (44)$$

This is again a contradiction to Eq. (42).

For both cases we have derived a contradiction. Thus Eq. (42) cannot hold, i.e., $T < 1$.

³The proof of this claim appears in the proof of Theorem 3 as well. The proof of Theorem 3 utilizes this lemma, thus we cannot directly utilize the conclusion of Theorem 3 in the proof of this lemma. Thus we restate that argument here.

Checklist

1. For all authors...
 - (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
 - (b) Did you describe the limitations of your work? [Yes] The limitations mainly come from the strong assumption of spherically symmetric distribution in Assumption 1, but numerous previous works employ the standard Gaussian data [8, 53, 45, 54, 32], which is a sub-case of ours.
 - (c) Did you discuss any potential negative societal impacts of your work? [N/A] The purpose of this work is to provide a better understanding of neural network optimization.
 - (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
 - (a) Did you state the full set of assumptions of all theoretical results? [Yes]
 - (b) Did you include complete proofs of all theoretical results? [Yes] See Appendix for detail.
3. If you ran experiments...
 - (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No] The code are proprietary, but we will display the code if required.
 - (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 5 for description.
 - (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]
 - (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] We do not use large datasets and the experiments is certainly simple and easy to follow.
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
 - (a) If your work uses existing assets, did you cite the creators? [N/A]
 - (b) Did you mention the license of the assets? [N/A]
 - (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
 - (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
 - (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
 - (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
 - (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
 - (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]