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# Secretary Ranking with Minimal Inversions

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## Abstract

We study a secretary problem which captures the task of ranking in online settings. We term this problem the *secretary ranking* problem: elements from an ordered set arrive in random order and instead of picking the maximum element, the algorithm is asked to assign a rank, or position, to each of the elements. The rank assigned is irrevocable and is given knowing only the pairwise comparisons with elements previously arrived. The goal is to minimize the distance of the rank produced to the true rank of the elements measured by the Kendall-Tau distance, which corresponds to the number of pairs that are inverted with respect to the true order.

Our main result is a matching upper and lower bound for the secretary ranking problem. We present an algorithm that ranks  $n$  elements with only  $O(n^{3/2})$  inversions in expectation, and show that any algorithm necessarily suffers  $\Omega(n^{3/2})$  inversions when there are  $n$  available positions. In terms of techniques, the analysis of our algorithm draws connections to linear probing in the hashing literature, while our lower bound result relies on a general anti-concentration bound for a generic balls and bins sampling process. We also consider the case where the number of positions  $m$  can be larger than the number of secretaries  $n$  and provide an improved bound by showing a connection of this problem with random binary trees.

## 1 Introduction

The secretary problem is one of the first problems studied in online algorithms—in fact, it was extensively studied much before the field of online algorithms even existed. It first appeared in print in 1960 as a recreational problem in Martin Gardner’s Mathematical Games column in Scientific American. In the subsequent decade it caught the attention of many of the eminent probabilist researchers like Lindley [Lin61], Dynkin [Dyn63], Chow et al. [CMRS64] and Gilbert and Mosteller [GM06] among others. In a very entertaining historical survey, Ferguson [Fer89] traces the origin of the secretary problem to much earlier: Cayley in 1875 and Kepler in 1613 pose questions in the same spirit as the secretary problem.

Secretary problem has been extended in numerous directions, see for example the surveys by Sakaguchi [Sak95] and Freeman [Fre83]. The problem has had an enormous influence in computer science and has provided some of the basic techniques in the field of online and approximation algorithms. Babaioff et al extended this problem to matroid set systems [BIK07] and Knapsack [BIKK07] and perhaps more importantly, show that the secretary problem is a natural tool for designing online auctions. In the last decade, the secretary problem has also been extended to posets [KLVV11], submodular systems [BHZ10], general set systems [Rub16], stable matchings [BEF<sup>+</sup>17], non-uniform arrivals [KKN15] and applied to optimal data sampling [GD09], design of prophet inequalities [AKW14, EHL17], crowdsourcing systems [SM13], pricing in online settings [CEFJ14], online linear programming [AWY14] and online ad allocation [FHK<sup>+</sup>10].

37 The (admittedly incomplete) list of extensions and applications in the last paragraph serves to  
 38 showcase that the secretary problem has traditionally been a vehicle for deriving connections between  
 39 different subfields of computer science and a testbed of new techniques.

40 **Ranking Secretaries.** We consider a natural variant of the secretary problem that captures ranking  
 41 from pairwise comparisons in online settings. In the *secretary ranking* problem, instead of selecting  
 42 the maximum element we are asked to *rank* each arriving element. In the process of deriving  
 43 the optimal algorithm for this problem, we uncover novel connections between ranking and the  
 44 technique of linear probing, which is one of the earliest techniques in the hashing literature studied  
 45 by Knuth [Knu63], and also the expected height of random binary trees.

46 In the traditional secretary problem a decision maker is trying to hire a secretary. There is a total  
 47 order over  $n$  secretaries and the goal of the algorithm is to hire the best secretary. The secretaries  
 48 are assumed to arrive in a random order and the algorithm can only observe the relative rank of  
 49 each secretary with respect to the previously interviewed ones. Once a secretary is interviewed,  
 50 the algorithm needs to decide whether to hire the current one or to irrevocably abandon the current  
 51 candidate and continue interviewing.

52 In our setting, there are  $m$  job positions and  $n$  secretaries. There is a known total order on positions.  
 53 Secretaries arrive in random order and, as before, we can only compare a secretary with previously  
 54 interviewed ones. In our version, all secretaries will be hired and the decision of the algorithm is in  
 55 which position to hire each secretary. Each position can be occupied by at most one secretary and  
 56 hiring decisions are irrevocable. Ideally, the algorithm will hire the best secretary in the best position,  
 57 the second best secretary in the second best position and so on. The loss incurred by the algorithm  
 58 corresponds to the pairs that are incorrectly ordered, i.e., pairs where a better secretary is hired in a  
 59 worse position.

60 We give two examples that illustrate scenarios where irrevocable ranking decisions occur online. The  
 61 first is in the context of task assignments. For concreteness, consider a consulting firm with teams of  
 62 different skill levels. Projects of different difficulty arrive in an online fashion and when a project  
 63 arrives, the firm needs to decide which team will execute. Of course, the most difficult projects should  
 64 go to the most skillful team. The second example is in the context of reward allocation. Consider a  
 65 university department that would like to assign the best scholarships available to the best students.  
 66 However, scholarships arrive one at a time and the school needs to decide which student is assigned  
 67 that scholarship knowing only the relative quality of the scholarships arrived so far.

## 68 1.1 Our Results and Techniques

69 The perhaps most natural case of the secretary ranking problem is when the numbers of positions  
 70 and secretaries are the same, i.e.  $m = n$ , which we call the dense case. The trivial algorithm that  
 71 assigns a random empty position for each arriving secretary incurs  $\Theta(n^2)$  cost, since each pair of  
 72 elements has probability  $1/2$  of being an inversion. On the other hand,  $\Omega(n)$  is a trivial lower bound  
 73 on the cost of any algorithm because nothing is known when the first element arrives. As such, there  
 74 is a linear gap between the costs of the trivial upper and lower bounds for this secretary ranking  
 75 problem. Our main result is an asymptotically tight upper and lower bound on the loss incurred by  
 76 the algorithms for the secretary ranking problem.

77 **Theorem.** *There is an algorithm for the secretary ranking problem that computes a ranking with*  
 78  *$\mathcal{O}(n^{3/2})$  inversions in expectation. Moreover, any algorithm for this problem makes  $\Omega(n^{3/2})$  inver-*  
 79 *sions in expectation.*

80 There are two challenges in designing an algorithm for secretary ranking. In earlier time steps, there  
 81 are only a small number of comparisons observed and these do not contain sufficient information  
 82 to estimate the true rank of the arriving elements. In later time steps, we observe a large number  
 83 of comparisons and using the randomness of elements arrival, the true rank of the elements can  
 84 be estimated well. However, the main difficulty is that at these time steps many of the positions  
 85 have already been assigned to some element arrived earlier and are hence not available. The first  
 86 information-theoretic challenge exacerbates this second issue. Previous bad placements might imply  
 87 that all the desired positions are unavailable for the current element, causing a large cost even for an  
 88 element whose true rank is estimated accurately.

89 The algorithm needs to handle these two opposing challenges simultaneously. The main idea behind  
 90 our algorithm is to estimate the rank of the current element using the observed comparisons and  
 91 then add some noise to these estimations to obtain additional randomness in the positions and avoid  
 92 positively correlated mistakes. We then assign the current element to the closest empty position to  
 93 this noisy estimated rank. The main technical interest is in the analysis of this algorithm. We draw a  
 94 connection to the analysis of linear probing in the hashing literature [Knu63] to argue that under this  
 95 extra noise, there often exists an empty position that is close to the estimated rank.

96 For the lower bound, we analyze the number of random pairwise comparisons needed to estimate  
 97 the rank of an element accurately. Such results are typically proven in the literature by using anti-  
 98 concentration inequalities. A main technical difficulty is that most of the existing anti-concentration  
 99 inequalities are for independent random variables while there is a correlation between the variables  
 100 we are considering. We prove, to the best of our knowledge, a new anti-concentration inequality for a  
 101 generic balls in bins problem that involves correlated sampling.

102 In the appendix, we study two additional cases of the secretary ranking problem. In the sparse case,  
 103 we wish to compute how large the number  $m$  of positions needs to be such that we incur no inversions.  
 104 Clearly for  $m = 2^{n+1} - 1$  it is possible to obtain zero inversions with probability 1 and for any  
 105 number less than that it is also clear that any algorithm needs to cause inversions with non-zero  
 106 probability. If we only want to achieve zero inversions with high probability, how large does  $m$  need  
 107 to be? By showing a connection between the secretary problem and random binary trees, we show  
 108 that for  $m \geq n^\alpha$  for  $\alpha \approx 2.998$  it is possible to design an algorithm that achieves zero inversion  
 109 with probability  $1 - o(1)$ . The constant  $\alpha$  here is obtained using the high probability bound on the  
 110 height of a random binary tree of  $n$  elements. Finally, we combine the algorithms for the dense and  
 111 sparse cases to obtain a general algorithm with a bound on the expected number of inversions which  
 112 smoothly interpolates between the bounds obtained for the dense and sparse cases.

## 113 1.2 Related Work

114 Our work is inserted in the vast line of literature on the secretary problem, which we briefly discussed  
 115 earlier. There has been a considerable amount of work on multiple-choice secretary problems  
 116 where, instead of the single best element, multiple elements can be chosen as they arrive online  
 117 [Kle05, BIKK07, BIK07, BHZ10, Rub16, KP09]. We note that in multiple-choice secretary problems,  
 118 the decision at arrival of an element is still binary, whereas in secretary ranking one of  $n$  positions  
 119 must be chosen. More closely related to our work is a paper of Babichenko et al. [BEF<sup>+</sup>17] where  
 120 elements that arrive must also be assigned to a position. However, the objective is different and the  
 121 goal, which uses a game-theoretic notion of stable matching, is to maximize the number of elements  
 122 that are not in a blocking pair. Gobel et al. [GKT15] also studied an online appointment scheduling  
 123 problem in which the goal is to assign starting dates to a set of jobs arriving online. The objective  
 124 here is again different from the secretary ranking problem and is to minimize the total weight time of  
 125 the jobs.

126 Another related line of work in machine learning is the well-known problem of learning to rank that  
 127 has been extensively studied in recent years (e.g. [BSR<sup>+</sup>05, CQL<sup>+</sup>07, BRL07, XLW<sup>+</sup>08]). Two im-  
 128 portant applications of this problem are search engines for document retrieval [L<sup>+</sup>09, RJ05, LXQ<sup>+</sup>07,  
 129 CXL<sup>+</sup>06, XL07] and collaborative filtering approaches to recommender systems [SLH10, SKB<sup>+</sup>12,  
 130 LY08, WRdVR08]. There has been significant interest recently in ranking from pairwise compar-  
 131 isons [FRPU94, BFSC<sup>+</sup>13, CS15, SW17, JKSO16, HSRW16, DKMR14, BMW16, AAK17]. To  
 132 the best of our knowledge, there has not been previous work on ranking from pairwise comparisons  
 133 in an online setting.

134 Finally, we also briefly discuss hashing, since our main technique is related to linear probing. Linear  
 135 probing is a classic implementation of hash tables and was first analyzed theoretically by Knuth  
 136 in 1963 [Knu63], in a report which is now regarded as the birth of algorithm analysis. Since then,  
 137 different variants of this problem mainly for hash functions with limited independence have been  
 138 considered in the literature [SS90, PPR07, PT10]. Reviewing the vast literature on this subject is  
 139 beyond the scope of our paper and we refer the interested reader to these papers for more details.

140 **Organization.** The remainder of the paper is organized as follows. In Section 2 we formalize the  
 141 secretary ranking problem. In Section 3, we present and analyze our algorithm. Section 4 is devoted  
 142 to showing the lower bound. Our results for the case where the number of position  $m$  is different

from the number of elements  $n$  appear in Appendix B and Appendix C. Missing proofs and standard concentration bounds are postponed to the appendix as well.

## 2 Problem Setup

In the secretary ranking problem, there are  $n$  elements  $a_1, \dots, a_n$  that arrive one at a time in an online manner and in a uniformly random order. There is a total ordering among the elements, but the algorithm has only access to pairwise comparisons among the elements that have already arrived. In other words, at time  $t$ , the algorithm only observes whether  $a_i < a_j$  for all  $i, j \leq t$ .

We define the rank function  $\text{rk} : \{a_1, \dots, a_n\} \rightarrow [n]$  as the true rank of the elements in the total order, i.e.,  $a_i < a_j$  iff  $\text{rk}(a_i) < \text{rk}(a_j)$ . Since the elements arrive uniformly at random,  $\text{rk}(\cdot)$  is a random permutation. Upon arrival of an element  $a_t$  at time step  $t$ , the algorithm must, irrevocably, place  $a_t$  in a position  $\pi(a_t) \in [n]$  that is not yet occupied, in the sense that for  $a_t \neq a_s$  we must have  $\pi(a_s) \neq \pi(a_t)$ . Since the main goal of the algorithm is to place the elements as to reflect the true rank as close as possible<sup>1</sup>, we refer to  $\pi(a_t)$  as the *learned rank* of  $a_t$ . The goal is to minimize the number of pairwise mistakes induced by the learned ranking compared to the true ranking. A pairwise mistake, or an inversion, is defined as a pair of elements  $a_i, a_j$  such that  $\text{rk}(a_i) < \text{rk}(a_j)$  according to the true underlying ranking but  $\pi(a_i) > \pi(a_j)$  according to the learned ranking.

The secretary ranking problem generalizes the secretary problem in the following sense: in the secretary problem, we are only interested in finding the element with the highest rank. However, in the secretary ranking problem, the goal is to assign a rank to every arrived element and construct a complete ranking of all elements. Similar to the secretary problem, we make the enabling assumption that the order of elements arrival is uniformly random.<sup>2</sup> We measure the cost of the algorithm in expectation over the randomness of both the arrival order of elements and the algorithm.

**Measures of sortedness.** We point out that the primary goal in the secretary ranking problem is to learn an ordering  $\pi$  of the input elements which is as close as possible to their sorted order. As such, the *cost* suffered by an algorithm is given by a *measure of sortedness* of  $\pi$  compared to the true ranking. There are various measures of sortedness studied in the literature depending on the application. Our choice of using the number of inversions, also known as *Kendall's tau* measure, as the cost of an algorithm is motivated by the importance of this measure and its close connection to other measures such as *Spearman's footrule* (see, e.g., Chapter 6B in [Dia88]).

For a mapping  $\pi : [n] \rightarrow [n]$ , Kendall's tau  $K(\pi)$  measures the number of inversions in  $\pi$ , i.e.:

$$K(\pi) := |\{(i, j); (\pi(a_i) - \pi(a_j))(\text{rk}(a_i) - \text{rk}(a_j)) < 0\}|.$$

Another important measure of sortedness is Spearman's footrule  $F(\pi)$  given by:  $F(\pi) := \sum_{i=1}^n |\text{rk}(a_i) - \pi(a_i)|$ , which corresponds to the summation of distances between the true rank of each element and its current position. A celebrated result of Diaconis and Graham [DG77] shows that these two measures are within a factor of two of each other, namely,  $K(\pi) \leq F(\pi) \leq 2 \cdot K(\pi)$ . We refer to this inequality as the DG inequality throughout the paper. Thus, up to a factor of two, the goals of minimizing the Kendall tau or Spearman's footrule distances are equivalent and, while the Kendall tau distance is used in the formulation of the problem, we also use the Spearman's footrule distance in the analysis.

## 3 The Algorithm

In this section, we describe and analyze an algorithm for the secretary ranking problem. Our main algorithmic result is the following theorem.

**Theorem 1.** *There exists an algorithm for the secretary ranking problem that incurs a cost of  $O(n\sqrt{n})$  in expectation.*

In Section 4, we show that this cost incurred by the algorithm is asymptotically optimal.

<sup>1</sup>In other words, hire the better secretaries in better positions.

<sup>2</sup>It is straightforward to verify that when the ordering is adversarial, any algorithm incurs the trivial cost of  $\Omega(n^2)$ . For completeness, a proof is provided in Appendix F.

### 3.1 Description of the Algorithm

The general approach behind the algorithm in Theorem 1 is as follows.

Upon the arrival of element  $a_t$  at time step  $t$ :

1. **Estimation step:** Estimate the true rank of the arrived element  $a_t$  using the *partial* comparisons seen so far.
2. **Assignment step:** Find the nearest currently unassigned rank to this estimate and let  $\pi(a_t)$  be this position.

We now describe the algorithm in more details. A natural way to estimate the rank of the  $t$ -th element in the estimation step is to compute the rank of this element with respect to the previous  $t - 1$  elements seen so far and then scale this number to obtain an estimate of the rank of this element between 1 and  $n$ . However, for our analysis of the assignment step, we need to tweak this approach slightly: instead of simply rescaling and rounding, we add perturbation to the estimated rank and then round its value. This gives a nice distribution of estimated ranks which is crucial for the analysis of the assignment step. The assignment step then simply assigns a learned rank to the element as close as possible to its estimated rank. We formalize the algorithm in Algorithm 1.

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#### ALGORITHM 1: Dense Ranking

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1 Input: a set of  $n$  positions, denoted here by  $[n]$ , and at most  $n$  online arrivals.
2 for any time step  $t \in [n]$  and element  $a_t$  do
3   Define  $r_t := |\{a_{t'} \mid a_{t'} < a_t \text{ and } t' < t\}|$ .
196 4   Sample  $x_t$  uniformly in the real interval  $[r_t \cdot \frac{n}{t}, (r_t + 1) \cdot \frac{n}{t}]$  and choose  $\tilde{rk}(a_t) = \lceil x_t \rceil$ .
5   Set the learned rank of  $a_t$  as  $\pi(a_t) = \arg \min_{i \in R} |i - \tilde{rk}(a_t)|$  and remove  $i$  from  $R$ .
6 end

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We briefly comment on the runtime of the algorithm. By using any self-balancing binary search tree—such as a red-black tree or an AVL tree—to store the ranking of the arrived elements as well as the set  $R$  of available ranks separately, Algorithm 1 is implementable in  $O(\log n)$  time for each step, so total  $O(n \log n)$  worst-case time.

We also note some similarity between this algorithm and linear probing in hashing. Linear probing is an approach to resolving collisions in hashing where, when a key is hashed to a non-empty cell, the closest neighboring cells are visited until an empty location is found for the key. The similarity is apparent to our assignment step which finds the nearest currently unassigned rank to the estimated rank of an element. The analysis of the assignment step follows similar ideas as the analysis for the linear probing hashing scheme.

### 3.2 The Analysis

The total number of inversions can be approximated within a factor of 2 by the Spearman's footrule. Therefore, we can write the cost of Algorithm 1 (up to a factor 2) as follows:

$$\sum_{t=1}^n |rk(a_t) - \pi(a_t)| \leq \sum_{t=1}^n |rk(a_t) - \tilde{rk}(a_t)| + \sum_{t=1}^n |\tilde{rk}(a_t) - \pi(a_t)|.$$

This basically breaks the cost of the algorithm in two parts: one is the cost incurred by the estimation step and the other one is the cost of the assignment step. Our analysis then consists of two main parts where each part bounds one of the terms in the RHS above. In particular, we first prove that given the partial comparisons seen so far, we can obtain a relatively good estimation to the rank of the arrived element, and then in the second part, we show that we can typically find an unassigned position in the close proximity of this estimated rank to assign to it. The following two lemmas capture each part separately. In both lemmas, the randomness in the expectation is taken over the random arrivals and the internal randomness of the algorithm:

216 **Lemma 3.1** (Estimation Cost). *In Algorithm 1*,  $\mathbb{E} \left[ \sum_{t=1}^n \left| \text{rk}(a_t) - \tilde{\text{rk}}(a_t) \right| \right] = O(n\sqrt{n})$ .

217 **Lemma 3.2** (Assignment Cost). *In Algorithm 1*,  $\mathbb{E} \left[ \sum_{t=1}^n \left| \tilde{\text{rk}}(a_t) - \pi(a_t) \right| \right] = O(n\sqrt{n})$ .

218 Theorem 1 then follows immediately from these two lemmas and Eq (3.2). The main part of the  
 219 argument is the analysis of the assignment cost, i.e., Lemma 3.2, and in particular its connection to  
 220 linear probing. The analysis for the estimation cost, i.e., Lemma 3.1, follows from standard Chernoff  
 221 bound arguments and is deferred to Appendix D.

**Assignment cost: proof of Lemma 3.2.** It is useful to think of sampling a random permutation in the following recursive way: given a random permutation over  $t - 1$  elements, it is possible to obtain a random permutation over  $t$  elements by inserting the  $t$ -th element in a uniformly random position between these  $t - 1$  elements. Formally, given  $\sigma : [t - 1] \rightarrow [t - 1]$ , if we sample a position  $i$  uniformly from  $[t]$  and generate permutation  $\sigma' : [t] \rightarrow [t]$  such that:

$$\sigma'(t') = \begin{cases} i & \text{if } t' = t \\ \sigma(t') & \text{if } t' < t \text{ and } \sigma'(t') < i \\ \sigma(t') + 1 & \text{if } t' < t \text{ and } \sigma'(t') > i \end{cases}$$

222 then  $\sigma'$  will be a random permutation over  $t$  elements. It is simple to see that just by fixing any  
 223 permutation and computing the probability of it being generated by this process.

Thinking about sampling the permutation in this way is very convenient for this analysis since at the  $t$ -th step of the process, the relative order of the first  $t$  elements is fixed (even though the true ranks can only be determined in the end). In that spirit, let us also define for a permutation  $\sigma : [t] \rightarrow [t]$  the event  $\mathcal{O}_\sigma$  that  $\sigma$  is the relative ordering of the first  $t$  elements:

$$\mathcal{O}_\sigma = \{a_{\sigma(1)} < a_{\sigma(2)} < \dots < a_{\sigma(t)}\}.$$

224 The following proposition asserts that the randomness of the arrival and the inner randomness of the  
 225 algorithm, ensures that the estimated ranks at each time step are chosen *uniformly at random* from all  
 226 possible ranks in  $[n]$ .

227 **Proposition 3.3.** *The values of  $\tilde{\text{rk}}(a_1), \dots, \tilde{\text{rk}}(a_n)$  are i.i.d and uniformly chosen from  $[n]$ .*

*Proof.* First let us show that for any fixed permutation  $\sigma$  over  $t - 1$  elements, the relative rank  $r_t$  defined in the algorithm is uniformly distributed in  $\{0, \dots, t - 1\}$ . In other words:

$$\Pr[r_t = i \mid \mathcal{O}_\sigma] = \frac{1}{t}, \quad \forall i \in \{0, \dots, t - 1\}.$$

228 Simply observe that there are exactly  $t$  permutations over  $t$  elements such that the permutation  
 229 induced in the first  $t - 1$  elements is  $\sigma$ . Since we are sampling a random permutation in this process,  
 230 each of these permutation are equally likely to happen. Moreover, since each permutation corresponds  
 231 to inserting the  $t$ -th element in one of the  $t$  positions, we obtain the bound.

232 Furthermore, since the probability of each value of  $r_t$  does not depend on the induced permutation  
 233  $\sigma$  over the first  $t - 1$  elements, then  $r_t$  is independent of  $\sigma$ . Since all the previous values  $r_{t'}$  are  
 234 completely determined by  $\sigma$ ,  $r_t$  is independent of all previous  $r_{t'}$  for  $t' < t$ .

235 Finally observe that if  $r_t$  is random from  $\{0, \dots, t - 1\}$ , then  $x_t$  is sampled at random from  $[0, n]$ , so  
 236  $\tilde{\text{rk}}(a_t)$  is sampled at random from  $[n]$ . Since for different values of  $t \in [n]$ , all  $r_t$  are independent, all  
 237 the values of  $\tilde{\text{rk}}(a_t)$  are also independent.  $\square$

238 Now that we established that  $\tilde{\text{rk}}(a_t)$  are independent and uniform, our next task is to bound how  
 239 far from the estimated rank we have to go in the assignment step, before we are able to assign a  
 240 learned rank to this element. This part of our analysis will be similar to the analysis of the linear  
 241 probing hashing scheme. If we are forced to let the learned rank of  $a_t$  be far away from  $\tilde{\text{rk}}(a_t)$ ,  
 242 say  $\left| \tilde{\text{rk}}(a_t) - \pi(a_t) \right| > k$ , then this necessarily means that all positions in the integer interval  
 243  $[\tilde{\text{rk}}(a_t) - k : \tilde{\text{rk}}(a_t) + k]$  must have already been assigned as a learned rank of some element. In the



following, we bound the probability of such an event happening for large values of  $k$  compared to the current time step  $t$ .

We say that the integer interval  $I = [i : i + s - 1]$  of size  $s$  is *popular* at time  $t$ , iff at least  $s$  elements  $a_{t'}$  among the  $t - 1$  elements that appear before the  $t$ -th element have estimated rank  $\tilde{\mathbf{rk}}(a_{t'}) \in I$ . Since by Proposition 3.3 every element has probability  $s/n$  of having estimated rank in  $I$  and the estimated ranks are independent, we can bound the probability that  $I$  is popular using a standard application of Chernoff bound (proof deferred to Appendix D).

**Claim 3.4.** *Let  $\alpha \geq 1$ , an interval of size  $s \geq 2\alpha \max\left(1, \left(\frac{t}{n-t}\right)^2\right)$  is popular at time  $t$  w.p.  $e^{-O(\alpha)}$ .*

We now use the above claim to bound the deviation between  $\tilde{\mathbf{rk}}(a_t)$  and  $\pi(a_t)$ . The following lemma is the key part of the argument.

**Lemma 3.5.** *For any  $t \leq n$ , we have  $\mathbb{E} \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| = O(\max\left(1, \left(\frac{t}{n-t}\right)^2\right))$ .*

*Proof.* Fix any  $\alpha \geq 1$ . We claim that, if the learned rank of  $a_t$  is a position which has distance at least  $k_\alpha = 4\alpha \cdot \max\left(1, \left(\frac{t}{n-t}\right)^2\right)$  from its estimated rank, then necessarily there exists an interval  $I$  of length at least  $2k_\alpha$  which contains  $\tilde{\mathbf{rk}}(a_t)$  and is popular.

Let us prove the above claim then. Let  $I$  be the shortest integer interval  $[a : b]$  which contains  $\tilde{\mathbf{rk}}(a_t)$  and moreover both positions  $a$  and  $b$  are not assigned to a learned rank by time  $t$  (by this definition,  $\pi(a_t)$  would be either  $a$  or  $b$ ). For  $\left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right|$  to be at least  $k_\alpha$ , the length of interval  $I$  needs to be at least  $2k_\alpha$ . But for  $I$  to have length at least  $2k_\alpha$ , we should have at least  $2k$  elements from  $a_1, \dots, a_{t-1}$  to have an estimated rank in  $I$ : this is simply because  $a$  and  $b$  are not yet assigned a rank by time  $t$  and hence any element  $a_{t'}$  which has estimated rank outside the interval  $I$  is never assigned a learned rank inside  $I$  (otherwise the assignment step should pick  $a$  or  $b$ , a contradiction).

We are now ready to finalize the proof. It is straightforward that in the above argument, it suffices to only consider the integer intervals  $[\tilde{\mathbf{rk}}(a_t) - k_\alpha : \tilde{\mathbf{rk}}(a_t) + k_\alpha]$  parametrized by the choice of  $\alpha \geq 1$ . By the above argument and Claim 3.4, for any  $\alpha \geq 1$ , we have,

$$\begin{aligned} \mathbb{E} \left[ \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| \right] &\leq \int_{\alpha=0}^{\infty} \Pr \left( \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| > k_\alpha \right) \cdot k_\alpha \cdot d\alpha \\ &\leq \int_{\alpha=0}^{\infty} \Pr \left( \text{Integer interval } [\tilde{\mathbf{rk}}(a_t) - k_\alpha : \tilde{\mathbf{rk}}(a_t) + k_\alpha] \text{ is popular} \right) \cdot k_\alpha \cdot d\alpha \\ &\stackrel{\text{Claim 3.4}}{\leq} O(\max\left(1, \left(\frac{t}{n-t}\right)^2\right)) \cdot \int_{\alpha=0}^{\infty} e^{-O(\alpha)} \cdot \alpha \cdot d\alpha \\ &= O(\max\left(1, \left(\frac{t}{n-t}\right)^2\right)). \end{aligned} \quad \square$$

We are now ready to finalize the proof of Lemma 3.2.

*Proof of Lemma 3.2.* We have,  $\mathbb{E} \left[ \sum_{t=1}^n \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| \right] = \sum_{t=1}^n \mathbb{E} \left[ \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| \right]$  by linearity of expectation. For any  $t < n/2$ , the maximum term in RHS of Lemma 3.5 is 1 and hence in this case, we have  $\mathbb{E} \left[ \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| \right] = O(1)$ . Thus, the contribution of the first  $n/2 - 1$  terms to the above summation is only  $O(n)$ . Also, when  $t > n - \sqrt{n}$ , we can simply write  $\mathbb{E} \left[ \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| \right] \leq n$  which is trivially true and hence the total contribution of these  $\sqrt{n}$  summands is also  $O(n\sqrt{n})$ . It remains to bound the total contribution of  $t \in [n/2, n - \sqrt{n}]$ . By Lemma 3.5,  $\sum_{t=n/2}^{n-\sqrt{n}} \mathbb{E} \left[ \left| \tilde{\mathbf{rk}}(a_t) - \pi(a_t) \right| \right] \leq O(1) \cdot \sum_{t=n/2}^{n-\sqrt{n}} \left(\frac{t}{n-t}\right)^2 = O(n\sqrt{n})$ , where the equality is by a simple calculation (see Proposition D.3 in Appendix D).  $\square$

## 4 A Tight Lower Bound

We complement the algorithmic result from the previous section by showing that the cost incurred by the algorithm is asymptotically optimal.

**Theorem 2.** *Any algorithm for the secretary ranking problem incurs  $\Omega(n\sqrt{n})$  cost in expectation.*

To prove Theorem 2, we first show that no deterministic algorithm can achieve better than  $O(n\sqrt{n})$  inversions and then use Yao's minimax principle to extend the lower bound to randomized algorithms (by simply fixing the randomness of the algorithm to obtain a deterministic one with the same performance over the particular distribution of the input).

The main ingredient of our proof of Theorem 2 is an anti-concentration bound for sampling without replacement which we cast as a balls in bins problem. We start by describing this balls in bin problem and prove the anti-concentration bound in Lemma 4.1. Lemma 4.2 then connects the problem of online ranking to the balls in bins problem.

To continue, we introduce some asymptotic notation that is helpful for readability. We write  $v = \Theta_1(n)$  if variable  $v$  is linear in  $n$ , but also smaller and bounded away from  $n$ , i.e.,  $v = cn$  for some constant  $c$  such that  $0 < c < 1$ .

**Lemma 4.1.** *Assume there are  $n$  balls in a bin,  $r$  of which are red and the remaining  $n - r$  are blue. Suppose  $t < \min(r, n - r)$  balls are drawn from the bin uniformly at random without replacement, and let  $\mathcal{E}_{k,t,r,n}$  be the event that  $k$  out of those  $t$  balls are red. Then, if  $r = \Theta_1(n)$  and  $t = \Theta_1(n)$ , for every  $k \in \{0, \dots, t\}$ :  $\Pr(\mathcal{E}_{k,t,r,n}) = O(1/\sqrt{n})$ .*

Our high level approach toward proving Lemma 4.1 is as follows:

1. We first use a counting argument to show that  $\Pr(\mathcal{E}_{k,t,r,n}) = \binom{r}{k} \binom{n-r}{t-k} / \binom{n}{t}$ .
2. We then use Stirling's approximation to show  $\binom{r}{k} \binom{n-r}{t-k} / \binom{n}{t} = O(n^{-1/2})$  for  $k = \lfloor \frac{tr}{n} \rfloor$ .
3. Finally, with a max. likelihood argument, we show that  $\arg \max_{k \in [n]} \binom{r}{k} \binom{n-r}{t-k} / \binom{n}{t} \approx \frac{tr}{n}$ .

By combining these, we have,  $\Pr(\mathcal{E}_{k,t,r,n}) \leq \max_{k \in [n]} \binom{r}{k} \binom{n-r}{t-k} / \binom{n}{t} \leq \binom{r}{k^*} \binom{n-r}{t-k^*} / \binom{n}{t}$  for  $k^* \approx \frac{tr}{n}$  (by the third step), which we bounded by  $O(n^{-1/2})$  (in the second step). The actual proof is however rather technical and is postponed to Appendix E.

The next lemma shows that upon arrival of  $a_t$ , any position has probability at least  $O(1/\sqrt{n})$  of being the correct rank for  $a_t$ , under some mild conditions. The proof of this lemma uses the previous anti-concentration bound for sampling without replacement by considering the elements smaller than  $a_t$  to be the red balls and the elements larger than  $a_t$  to be the blue balls. For  $a_t$  to have rank  $r$  and be the  $k$ th element in the ranking so far, the first  $t - 1$  elements previously observed must contain  $k - 1$  red balls out of the  $r - 1$  red balls and  $t - k$  blue balls out of the  $n - r$  blue balls.

**Lemma 4.2.** *Fix any permutation  $\sigma$  of  $[t]$  and let  $\mathcal{O}_\sigma$  denote the event that  $a_{\sigma(1)} < a_{\sigma(2)} < \dots < a_{\sigma(t)}$ . If  $\sigma(k) = t$ ,  $k = \Theta_1(t)$  and  $t = \Theta_1(n)$  then for any  $r$ :  $\Pr(\text{rk}(a_t) = r \mid \mathcal{O}_\sigma) = O(1/\sqrt{n})$ .*

*Proof.* Define  $\mathcal{E}_k$  as the event that “ $a_t$  is the  $k$ -th smallest element in  $a_1, \dots, a_t$ ”. We first have,  $\Pr(\text{rk}(a_t) = r \mid \mathcal{O}_\sigma) = \Pr(\text{rk}(a_t) = r \mid \mathcal{E}_k)$ . This is simply because  $\text{rk}(a_t)$  is only a function of the pairwise comparisons of  $a_t$  with other elements and does not depend on the ordering of the remaining elements between themselves. Moreover,

$$\Pr(\text{rk}(a_t) = r \mid \mathcal{E}_k) = \Pr(\mathcal{E}_k \mid \text{rk}(a_t) = r) \cdot \frac{\Pr(\text{rk}(a_t) = r)}{\Pr(\mathcal{E}_k)} = \Pr(\mathcal{E}_k \mid \text{rk}(a_t) = r) \cdot \frac{t}{n}$$

since  $a_t$  is randomly partitioned across the  $[n]$  elements. Notice now that conditioned on  $\text{rk}(a_t) = r$ , the event  $\mathcal{E}_k$  is exactly the event  $\mathcal{E}_{k-1,t-1,r-1,n-1}$  in the sampling without replacement process defined in Lemma 4.1. The  $n - 1$  balls are all the elements but  $a_t$ , the  $r - 1$  red balls correspond to elements smaller than  $a_t$ , the  $n - r$  blue balls to elements larger than  $a_t$ , and  $t - 1$  balls drawn are the elements arrived before  $a_t$ . Finally, observe that  $\Pr(r < k \mid \mathcal{E}_k) = 0$ , so for  $r < k$ , the bound holds trivially. In the remaining cases,  $r = \Theta_1(n)$  and we use the bound in Lemma 4.1 with  $t/n = \Theta(1)$ .  $\square$



322 Using the previous lemma, we can lower bound the cost due to the  $t$ -th element. Fix any deterministic  
 323 algorithm  $\mathcal{A}$  for the online ranking problem. Recall that  $\pi(a_t)$  denotes the learned rank of the item  
 324  $a_t$  arriving in the  $t$ -th time step. For any time step  $t \in [n]$ , we use  $\text{cost}_{\mathcal{A}}(t)$  to denote the cost  
 325 incurred by the algorithm  $\mathcal{A}$  in positioning the item  $a_t$ . More formally, if  $\text{rk}(a_t) = i$ , we have  
 326  $\text{cost}_{\mathcal{A}}(t) := |i - \pi(a_t)|$ . Theorem 2 then follows by Yao's minimax principle principle and the  
 327 following lemma, whose proof appears in the appendix

328 **Lemma 4.3.** *Fix any deterministic algorithm  $\mathcal{A}$ . For any  $t = \Theta_1(n)$ ,  $\mathbb{E}[\text{cost}_{\mathcal{A}}(t)] = \Omega(\sqrt{n})$ .*

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## A Useful Concentration of Measure Inequalities

We use the following two standard versions of Chernoff bound (see, e.g., [DP09]) throughout.

**Proposition A.1** (Multiplicative Chernoff bound). *Let  $X_1, \dots, X_n$  be  $n$  independent random variables taking values in  $[0, 1]$  and let  $X := \sum_{i=1}^n X_i$ . Then, for any  $\varepsilon \in (0, 1]$ ,*

$$\Pr(X \geq (1 + \varepsilon) \cdot \mathbb{E}[X]) \leq \exp(-2\varepsilon^2 \cdot \mathbb{E}[X]).$$

**Proposition A.2** (Additive Chernoff bound). *Let  $X_1, \dots, X_n$  be  $n$  independent random variables taking values in  $[0, 1]$  and let  $X := \sum_{i=1}^n X_i$ . Then,*

$$\Pr(|X - \mathbb{E}[X]| > t) \leq 2 \cdot \exp\left(-\frac{2t^2}{n}\right).$$

Moreover, if  $X_1, \dots, X_n$  are negatively correlated (i.e.  $\Pr[X_i = 1, \forall i \in S] \leq \prod_{i \in S} \Pr[X_i = 1]$  for all  $S \subseteq [n]$ ), then the upper tail holds:  $\Pr(X - \mathbb{E}[X] > t) \leq \exp\left(-\frac{2t^2}{n}\right)$ .

Moreover, in the above setting, if  $X$  comes from a sampling *with* replacement process, then the inequality holds for both upper and lower tails. For sampling without replacement, we refer to Serfling [Ser74] for a complete discussion and for Chernoff bounds for negatively correlated random variables see [PS97].

**Proposition A.3** (Chernoff bound for sampling without replacement). *Consider an urn with  $a \geq b$  red and blue balls. Draw  $b$  balls uniformly from the urn without replacement and let  $X$  be the number of red balls drawn, then the two sided bound holds:  $\Pr(|X - \mathbb{E}[X]| > t) \leq 2 \cdot \exp\left(-\frac{2t^2}{b}\right)$ .*

*Proof.* If  $X_i$  is the event that the  $i$ -th ball is red, then since  $X_i$  are negatively correlated, the upper tail Chernoff bound of  $X = \sum_i X_i$  holds. Now, let  $Y_i = 1 - X_i$  be the probability that the  $i$ -th ball is blue and  $Y = \sum_i Y_i$ . The upper tail for  $Y$  correspond to the lower tail for  $X$ , i.e.:  $\Pr(X - \mathbb{E}[X] < -t) = \Pr(Y - \mathbb{E}[Y] > t) \leq \exp\left(-\frac{2t^2}{b}\right)$ .  $\square$

## B Sparse Secretary Ranking

In this section, we consider the special case where the number of positions is very large, which we call sparse secretary ranking. In the extreme when  $m \geq 2^{n+1} - 1$  it is possible to assign a position to each secretary without ever incurring a mistake. To do that, build a complete binary tree of height  $n$  and associate each position in  $[m]$  with a node (both internal and leaf) of the binary tree such that the order of the positions corresponds to the pre-order induced by the binary tree (see figure B). Once the elements arrive in an online fashion, insert them in the binary tree and allocate them in the corresponding position.

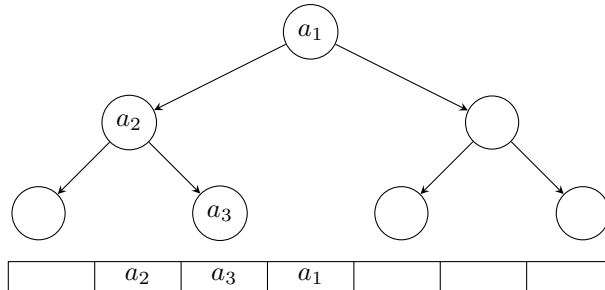


Figure 1: Illustration of the binary tree algorithm for  $m = 7$  and order  $a_2 < a_3 < a_1$ .

We note that the above algorithm works for any order of arrival. If the elements arrive in random order, it is possible to obtain zero inversions with high probability for an exponentially smaller value of  $m$ . The idea is very similar to the one outlined above. Let  $H_n$  be a random variable corresponding to the height of a binary tree built from  $n$  elements in random order. Reed [Ree03] shows that



524  $\mathbb{E}[H_n] = \alpha \ln(n)$ ,  $\text{Var}[H_n] = O(1)$  where  $\alpha$  is the solution of the equation  $\alpha \ln(2e/\alpha) = 1$  which is  
525  $\alpha \approx 4.31107$ .

526 Since the arrival order of secretaries is uniformly random, the binary tree algorithm won't touch any  
527 node with height more than  $\bar{h} = \lceil (\alpha + O(\epsilon)) \ln(n) \rceil$  with probability  $1 - o(1)$ . This observation  
528 allows us to define an algorithm that obtains zero inversions with probability  $1 - o(1)$ . If  $m \geq$   
529  $2^{\bar{h}+1} - 1 = \Omega(n^{2.998+\epsilon})$ , we can build a binary tree with height  $\bar{h}$  and associate each node of the  
530 tree to a position. Once the elements arrive, allocate the item in the corresponding position. If an  
531 item is added to the tree with height larger than  $\bar{h}$ , start allocating the items arbitrarily.

532 **Theorem 3.** *If  $m \geq n^{2.988+\epsilon}$  then the algorithm that allocates according to a binary tree incurs zero*  
533 *inversions with probability  $1 - o(1)$ .*

Devroye [Dev86] bounds the tail of the distribution of  $H_n$  as follows:

$$\Pr[H_n \geq k \cdot \ln n] \leq \frac{1}{n} \cdot \left(\frac{2e}{k}\right)^{k \cdot \ln n}$$

534 for  $k > 2$ . In particular:  $\Pr[H_n \geq 6.3619 \cdot \ln n] \leq 1/n^2$ . Adapting the analysis above, we can show  
535 that for  $m \geq 4.41$  (where  $4.41 = 6.3619 \cdot \ln(2)$ ) the algorithm incurs less than one inversion in  
536 expectation.

537 **Corollary 4.** *If  $m \geq \Omega(n^{4.41})$  then the algorithm that allocates according to a binary tree incurs*  
538  *$O(1)$  inversion in expectation.*

## 539 C General Secretary Ranking

540 In the general case, we combine the ideas for the sparse and dense case to obtain an algorithm  
541 interpolating both cases. As described in Algorithm 2, we construct a complete binary search tree of  
542 height  $h$  and associating one position for each internal node, but for the leaves we associate a block  
543 of  $w = m/2^h - 1$  positions (see Figure 2). If we insert an element in a leaf, we allocate according  
544 to an instance of the dense ranking algorithm. By that we mean that the algorithm pretends that  
545 the elements allocated to that leaf are an isolated instance of dense ranking with  $w$  elements and  
546  $w$  positions. We will set  $h$  such that in expectation there only  $w$  elements in each leaf with high  
547 probability. If at some point more than  $w$  elements are placed in any given leaf, the algorithm starts  
548 allocating arbitrarily.

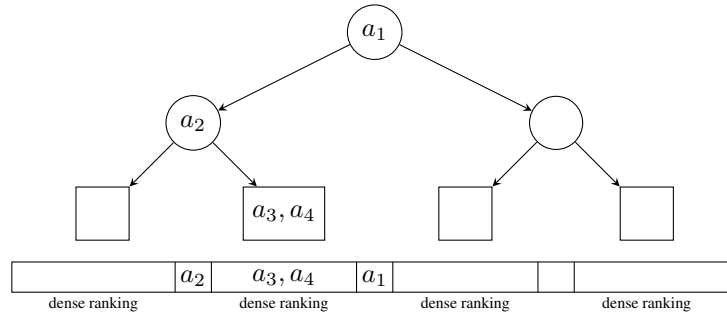


Figure 2: Illustration of the general algorithm (Algorithm 2) for order  $a_2 < a_3 < a_4 < a_1$ . The leaves are associated with blocks of  $w$  consecutive positions and internal nodes are associated with a single position. Elements  $a_3$  and  $a_4$  are associated with the same leaf and therefore we place them in a block of  $w$  positions as we would in a dense ranking problem with  $w$  arrivals and  $w$  positions.

---

**ALGORITHM 2:** General secretary ranking

---

```

1 Input: a set of  $m$  positions, at most  $n$  online arrivals and a height  $h$ .
2 Construct a complete binary search tree  $T$  of height  $h$  and associate one position for each internal node, and
  a block of  $w = m/2^h - 1$  positions for each leaf such that the order of the positions corresponds to the
  pre-order induced by the binary tree
3 for any time step  $t \in [n]$  and element  $a_t$  do
4   Insert  $a_t$  in the tree  $T$ 
5   if  $a_t$  reaches an empty internal node then
549   | Place  $a_t$  in the position corresponding to this internal node
6   end
7   else
8   | Place  $a_t$  according to an instance of the dense ranking algorithm (Algorithm 1) over the block of
9   | positions corresponding to the leaf reached by  $a_t$ . If there are no position available in that block,
    place  $a_t$  arbitrarily
10  end
11 end

```

---

For stating our main theorem and its proof, it is convenient to define the functions:

$$f(\alpha) = \frac{\alpha \ln(2) - 1}{1 - 2\alpha \ln(2e/\alpha)} \quad g(\alpha) = \frac{1}{1 - 2\alpha \ln(2e/\alpha)}$$

defined in the interval  $(\alpha_0, \infty)$  where  $\alpha_0 \approx 4.910$  is the solution to the equation  $1 - 2\alpha_0 \ln(2e/\alpha_0) = 0$ . Both functions are monotone decreasing from  $+\infty$  (when  $\alpha = \alpha_0$ ) to zero (when  $\alpha \rightarrow \infty$ ). We are now ready to state our main theorem:

**Theorem 5.** Assume  $m \geq 10n \log n$  and let  $\alpha \in (\alpha_0, \infty)$  be the solution to  $\frac{m}{9n \log n} = n^{f(\alpha)}$ , then the expected number of inversions of the general secretary ranking algorithm with  $h = \alpha \ln(n^{g(\alpha)})$  is  $\tilde{O}(n^{1.5-0.5g(\alpha)})$ .

We note that the algorithm smoothly interpolated between the two cases previously analyzed. When  $m = n \log(n)$  then  $\alpha \rightarrow \infty$ , so  $g(\alpha) \rightarrow 0$  and the bound on the theorem becomes  $\tilde{O}(n^{1.5})$ . In the other extreme, when  $m \rightarrow \infty$ , then  $\alpha \rightarrow \alpha_0$  and therefore  $g(\alpha) \rightarrow \infty$ , so the bound on the number of inversions becomes  $O(1)$ .

*Proof.* Let  $H_t$  be the height of the binary tree formed by the first  $t$  elements. By Devroye's bound [Dev86], the probability that a random binary tree formed by the first  $t := n^{g(\alpha)}$  elements has height more than  $h = \alpha \ln(t)$  is

$$\Pr[H_t \geq h] \leq \frac{1}{t} (2e/\alpha)^{\alpha \ln t} = t^{\alpha \ln(2e/\alpha) - 1}.$$

In case this event happens, we will use the trivial bound of  $O(n^2)$  on the number of inversions, which will contribute

$$n^2 t^{\alpha \ln(2e/\alpha) - 1} = n^{1.5 - 0.5/(1 - 2\alpha \ln(2e/\alpha))} = n^{1.5 - 0.5g(\alpha)}$$

to the expectation. From this point on, we consider the remaining event that  $H_t < h$ .

Next, we condition on the first  $t$  elements that we denote  $b_1, \dots, b_t$  such that  $b_1 < \dots < b_t$ . We note that for each remaining element  $a_i$ ,  $i > t$ , we have  $b_j < a_i < b_{j+1}$  with probability  $1/(t+1)$  for all  $j \in [t]$ . Since  $b_1, \dots, b_t$  are all placed in positions corresponding to internal nodes, each element has at most probability  $1/t$  of hitting any of the dense-ranking instances. Thus, each dense ranking instance receives at most  $n/t$  elements in expectation, and by a standard application of the Chernoff bound, the probability that a dense ranking instance sees more than  $9(n/t) \log n$  elements is  $n^{-3}$ . If this is the case for some dense ranking instance, we again use the  $n^2$  trivial bound, which contributes at most 1 to the expected number of inversions. For the remainder of the proof, we assume that each dense ranking instance gets at most  $9(n/t) \log n$  elements.

Next, note that the size of each block is

$$w = \frac{m}{2^h} - 1 = \frac{m}{t^{\alpha \ln(2)}} - 1 \geq 9(n/t) \log n$$

where the last equality is by definition of  $t$ . Thus, no more than  $w$  elements are inserted in any leaf.

Let  $v_i$  is the number of elements in each of the dense rank instances. We note that within the elements in each dense ranking block the arrival order is random, so we can apply the bound from Section 3 and obtain by Theorem 1 that the total expected cost from the inversions caused by dense rank is at most

$$\sum_i O(v_i^{1.5}) \leq \tilde{O}(t \cdot (n/t)^{1.5}) = \tilde{O}(n^2 t^{\alpha \ln(2e/\alpha) - 1}) = \tilde{O}(n^{1.5 - 0.5g(\alpha)})$$

since  $\sum_i v_i = n$  and  $v_i \leq (n/t) \log(n)$ . By the construction there are no inversions between elements inserted in different leaves and between an element inserted in an internal node and any other element. Summing the expected number of mistakes from the events  $H_t \geq h$  and  $H_t < h$ , we get the bound in the statement of the theorem.  $\square$

## D Missing Analysis from Section 3

**Estimation Cost: Proof of Lemma 3.1.** We begin with the following useful proposition.

**Proposition D.1.** *If  $1 < t \leq n$  and  $0 \leq r \leq t - 1$ , then  $r \cdot \left(\frac{n}{t}\right) \leq r \cdot \left(\frac{n-1}{t-1}\right) \leq (r+1) \cdot \left(\frac{n}{t}\right)$ .*

*Proof.*  $0 \leq r \left(\frac{n-1}{t-1} - \frac{n}{t}\right) = r \frac{n-t}{t(t-1)} \leq (t-1) \frac{n-t}{t(t-1)} \leq \frac{n}{t}$ .  $\square$

The correctness of the estimation step in our algorithm relies on the following proposition that bounds the probability of the deviation between the estimated rank and the true rank of each element (depending on the time step it arrives). The proof uses the Chernoff bound for sampling without replacement.

**Proposition D.2.** *For any  $t > 1$  and any  $\alpha \geq 0$ ,  $\Pr\left(\left|\text{rk}(a_t) - \tilde{\text{rk}}(a_t)\right| \geq 1 + \frac{n}{t} + \alpha \cdot \frac{n-1}{\sqrt{t-1}}\right) \leq e^{-\Omega(\alpha^2)}$ .*

*Proof.* Fix any  $t \in [n]$  and element  $a_t$  and recall that  $\text{rk}(a_t)$  denotes the true rank of  $a_t$ . Conditioned on a fixed value for the rank of  $a_t$ , the distribution of the number of elements  $r_t$  that arrived before  $a_t$  and have a smaller rank is equivalent to a sampling without replacement process of  $t-1$  balls where the urn has  $\text{rk}(a_t) - 1$  red balls and  $n - \text{rk}(a_t)$  blue balls (and the goal is to count the number of red balls). As such  $\mathbb{E}[r_t] = \frac{\text{rk}(a_t)-1}{n-1}$  and by the Chernoff bound for sampling without replacement (Proposition A.3 with  $a = n$  and  $b = t-1$ ), we have:

$$\Pr\left(\left|r_t - \mathbb{E}[r_t]\right| \geq \alpha\sqrt{t-1}\right) \leq 2 \cdot \exp\left(-\frac{2(\alpha\sqrt{t-1})^2}{t-1}\right) = e^{-\Omega(\alpha^2)}.$$

We now argue that

$$\Pr\left(\left|\text{rk}(a_t) - \tilde{\text{rk}}(a_t)\right| \geq 1 + \frac{n}{t} + \alpha \cdot \frac{n-1}{\sqrt{t-1}}\right) \leq \Pr\left(\left|r_t - \mathbb{E}[r_t]\right| \geq \alpha\sqrt{t-1}\right).$$

which finalizes the proof by the bound in above equation.

To see this, note that,

$$\alpha \frac{n-1}{\sqrt{t-1}} \geq \left|\frac{n-1}{t-1} r_t - \text{rk}(a_t)\right| \geq |x_t - \text{rk}(a_t)| - \frac{n}{t} \geq \left|\tilde{\text{rk}}(a_t) - \text{rk}(a_t)\right| - 1 - \frac{n}{t}$$

The first inequality follows from substituting the expectation in  $|r_t - \mathbb{E}[r_t]| \geq \alpha\sqrt{t-1}$  and multiplying the whole expression by  $(n-1)/(t-1)$ . The second inequality just follows from the fact that both the variable  $x_t$  (defined in step 4 of Algorithm 2) and  $\frac{n-1}{t-1} r_t$  are in the interval  $[\frac{n}{t} r_t, \frac{n}{t} (r_t + 1)]$ . The fact that  $x_t$  is in this interval comes directly from its definition in the algorithm and the fact that  $\frac{n-1}{t-1} r_t$  is in the interval is by a simple calculation (see Proposition D.1 in Appendix D). The last inequality follows from the fact that  $\tilde{\text{rk}}(a_t) = \lceil x_t \rceil$ .  $\square$

600 We are now ready to prove Lemma 3.1.

601 *Proof of Lemma 3.1.* Fix any  $t > 1$ ; we have,

$$\begin{aligned} \mathbb{E} \left[ \left| \text{rk}(a_t) - \tilde{\text{rk}}(a_t) \right| - 1 - \frac{n}{t} \right] &\leq \int_{\alpha=0}^{\infty} \Pr \left( \left| \text{rk}(a_t) - \tilde{\text{rk}}(a_t) \right| - 1 - \frac{n}{t} \geq \alpha \cdot \frac{n-1}{\sqrt{t-1}} \right) \cdot \frac{n-1}{\sqrt{t-1}} \cdot d\alpha \\ &\leq \frac{n-1}{\sqrt{t-1}} \cdot \int_{\alpha=0}^{\infty} e^{-\Omega(\alpha^2)} \cdot d\alpha = O\left(\frac{n}{\sqrt{t}}\right). \quad (\text{by Proposition D.2}) \end{aligned}$$

602 Hence, using the trivial bound for  $t = 1$  and the bound above for  $t > 1$  we conclude that:

$$\mathbb{E} \left[ \sum_{t=1}^n \left| \text{rk}(a_t) - \tilde{\text{rk}}(a_t) \right| \right] = \sum_{t=1}^n \mathbb{E} \left[ \left| \text{rk}(a_t) - \tilde{\text{rk}}(a_t) \right| \right] = \sum_{t=1}^n O\left(\frac{n}{t} + \frac{n}{\sqrt{t}}\right) = O(n\sqrt{n})$$

603 **Missing analysis for Lemma 3.2.**

604 **Claim 3.4.** Let  $\alpha \geq 1$ , an interval of size  $s \geq 2\alpha \max\left(1, \left(\frac{t}{n-t}\right)^2\right)$  is popular at time  $t$  w.p.  $e^{-O(\alpha)}$ .

605 *Proof.* The proof follows directly from the Chernoff bound in Proposition A.1. For  $t' \in [t]$ , let  $X_{t'}$   
606 be the event that  $\tilde{\text{rk}}(a_{t'}) \in I$  and  $X = \sum_{t'=1}^t X_{t'}$ , then setting  $\epsilon = \min(1, \frac{n-t}{t})$  we have that:

$$\begin{aligned} \Pr(I \text{ is popular}) &= \Pr(X \geq s) \leq \Pr(X > (1 + \epsilon) \cdot \epsilon \cdot \mathbb{E}[X]) \\ &\leq \exp\left(-\frac{\epsilon^2 \cdot \mathbb{E}[X]}{2}\right) = e^{-O(\alpha)} \end{aligned}$$

607 as  $\mathbb{E}[X] = s \cdot t/n$ . □

608 **Proposition D.3.** For any integer  $n > 0$ ,  $\sum_{t=1}^{n-\sqrt{n}} \left(\frac{t}{n-t}\right)^2 = O(n\sqrt{n})$ .

609 *Proof.* By defining  $k = n - t$ , we have,

$$\sum_{t=1}^{n-\sqrt{n}} \left(\frac{t}{n-t}\right)^2 = \sum_{k=\sqrt{n}}^{n-1} \left(\frac{n-k}{k}\right)^2 \leq \sum_{k=\sqrt{n}}^{n-1} \left(\frac{n}{k}\right)^2$$

610 For  $i \in [\sqrt{n}]$ , define  $K_i := \{k \mid i \cdot \sqrt{n} \leq k < (i+1) \cdot \sqrt{n}\}$ . For any  $k \in K_i$ , we have,  $\frac{n}{k} \leq \frac{\sqrt{n}}{i}$ .  
611 As such, we can write,

$$\begin{aligned} \sum_{k=\sqrt{n}}^{n-1} \left(\frac{n}{k}\right)^2 &= \sum_{i=1}^{\sqrt{n}} \sum_{k \in K_i} \left(\frac{n}{k}\right)^2 \leq \sum_{i=1}^{\sqrt{n}} \sum_{k \in K_i} \left(\frac{\sqrt{n}}{i}\right)^2 \\ &\leq \sum_{i=1}^{\sqrt{n}} n \cdot |K_i| \cdot \frac{1}{i^2} \leq n\sqrt{n} \cdot \sum_{i=1}^{\sqrt{n}} \frac{1}{i^2} = O(n\sqrt{n}) \end{aligned}$$

612 as the series  $\sum_i \frac{1}{i^2}$  is a converging series. □

## 613 E Anti-Concentration for Sampling Without Replacement

614 We prove Lemma 4.1 restated here for convenience.

615 **Lemma** (Restatement of Lemma 4.1). Assume there are  $n$  balls in a bin,  $r$  of which are red and the  
616 remaining  $n - r$  are blue. Suppose  $t < \min(r, n - r)$  balls are drawn from the bin uniformly at  
617 random without replacement, and let  $\mathcal{E}_{k,t,r,n}$  be the event that  $k$  out of those  $t$  balls are red. Then, if  
618  $r = \Theta_1(n)$  and  $t = \Theta_1(n)$ , for every  $k \in \{0, \dots, t\}$ :  $\Pr(\mathcal{E}_{k,t,r,n}) = O(1/\sqrt{n})$ .

619 To prove Lemma 4.1, we will describe the sampling without replacement process explicitly and  
620 bound the relevant probabilities.

**Proposition E.1.** Let  $0 < c < 1$  be a constant. Then:

$$\binom{n}{cn} = \Theta(n^{-1/2} c^{-(cn+1/2)} (1-c)^{-((1-c)n+1/2)})$$

The notation  $y = \Theta(x)$  in the lemma statement means that there are universal constants  $0 < \underline{\alpha} < \bar{\alpha}$  independent of  $c$  and  $n$  such that  $\underline{\alpha} \cdot x \leq y \leq \bar{\alpha} \cdot x$ . The proof is based on the following version of Stirling's approximation:  $\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e n^{n+\frac{1}{2}} e^{-n}$ . which can be written in our notation as:  $n! = \Theta(n^{n+\frac{1}{2}} e^{-n})$ . The proof of the previous lemmas follows from just expanding the factorials in the definition of the binomial:

*Proof.* Observe that

$$\binom{n}{cn} = \frac{n!}{(cn)!((1-c)n)!} = \Theta\left(\frac{n^{n+\frac{1}{2}} e^{-n}}{(cn)^{cn+\frac{1}{2}} e^{-cn} ((1-c)n)^{(1-c)n+\frac{1}{2}} e^{-((1-c)n}}}\right)$$

The statement follows from simplifying the right hand side.  $\square$

**Lemma E.2.** Assume that  $r = \Theta_1(n)$  and  $t = \Theta_1(n)$  and  $t \leq \min(r, n-r)$ , then for  $k = \lfloor rt/n \rfloor$ , we have

$$\frac{\binom{r}{k} \cdot \binom{n-r}{t-k}}{\binom{n}{t}} = \mathcal{O}(1/\sqrt{n}).$$

*Proof.* Start by writing  $r = c_r \cdot n$  and  $t = c_t \cdot n$  for  $0 < c_r, c_t < 1$ . It will be convenient to assume that  $k = rt/n$  is an integer (if not and we need to apply floors, the exact same proof work by keeping track of the errors introduced by floor). Then we can write: First, note that

$$\frac{\binom{r}{k} \cdot \binom{n-r}{t-k}}{\binom{n}{t}} = \frac{\binom{c_r n}{c_t c_r n} \cdot \binom{(1-c_r)n}{(1-c_r)c_t n}}{\binom{n}{c_t n}}$$

We can now apply the approximation in Proposition E.1 obtaining:

$$\Theta\left(\frac{n^{1/2} c_t^{c_t n + \frac{1}{2}} (1-c_t)^{(1-c_t)n + \frac{1}{2}}}{(c_r n)^{1/2} c_t^{c_t(c_r n) + \frac{1}{2}} (1-c_t)^{(1-c_t)(c_r n) + \frac{1}{2}} ((1-c_r)n)^{1/2} c_t^{c_t((1-c_r)n) + \frac{1}{2}} (1-c_t)^{(1-c_t)((1-c_r)n) + \frac{1}{2}}}\right)$$

Simplifying this expressoin, we get:  $\Theta\left((nc_t(1-c_t)c_r(1-c_r))^{-1/2}\right) = \Theta_1(1/\sqrt{n})$ .  $\square$

**Lemma E.3.** Fix any  $r, t, n$  such that  $r, t \leq n$ . Then,

$$\arg \max_{k \in [n]} \frac{\binom{r}{k} \cdot \binom{n-r}{t-k}}{\binom{n}{t}} = \left\lfloor t \cdot \frac{r}{n} \right\rfloor \text{ or } \left\lceil t \cdot \frac{r}{n} \right\rceil.$$

*Proof.* The proof is again simpler if we assume  $k = tr/n$  is an integer. If not, the same argument works controlling the errors. In that case, let  $k_1 = tr/n + i$  and  $k_2 = tr/n + i + 1$  and as before, let  $r = c_r n$  and  $t = c_t n$ . Note that

$$\frac{\frac{\binom{r}{k_1} \cdot \binom{n-r}{t-k_1}}{\binom{n}{t}}}{\frac{\binom{r}{k_2} \cdot \binom{n-r}{t-k_2}}{\binom{n}{t}}} = \frac{\binom{r}{k_1} \cdot \binom{n-r}{t-k_1}}{\binom{r}{k_2} \cdot \binom{n-r}{t-k_2}} = \frac{\binom{c_r n}{c_t c_r n + i} \cdot \binom{(1-c_r)n}{(1-c_r)c_t n - i}}{\binom{c_r n}{c_t c_r n + i + 1} \cdot \binom{(1-c_r)n}{(1-c_r)c_t n - i - 1}} = \frac{(c_t c_r n + i + 1) \cdot ((1-c_t)(1-c_r)n + i + 1)}{((1-c_t)c_r n - i) \cdot ((1-c_r)c_t n - i)}$$

If  $i \geq 0$ , then the last term is at least  $\frac{(c_t c_r n) \cdot ((1-c_t)(1-c_r)n)}{((1-c_t)c_r n) \cdot ((1-c_r)c_t n)}$  which is greater than one. If  $i \leq -1$ , then the last term is  $\frac{(c_t c_r n) \cdot ((1-c_t)(1-c_r)n)}{((1-c_t)c_r n) \cdot ((1-c_r)c_t n)}$  which is smaller than one.

Thus,  $\frac{\binom{r}{k} \cdot \binom{n-r}{t-k}}{\binom{n}{t}}$  is increasing as  $k$  increases up to  $tr/n$  and then decreases. Thus, the maximum is reached at  $tr/n$ .  $\square$

*Proof of Lemma 4.1.* We first use a simple counting argument to obtain an expression for  $\Pr(\mathcal{E}_{k,t,r,n})$  as a ratio of binomial coefficients. We note that there are  $\binom{r}{k}$  collections of  $k$  red balls,  $\binom{n-r}{t-k}$  collections of  $t-k$  blue balls, and that the total number of collections of  $t$  balls is  $\binom{n}{t}$ . Since the  $t$  balls are drawn uniformly at random without replacement, we get

$$\Pr(\mathcal{E}_{k,t,r,n}) = \frac{\binom{r}{k} \cdot \binom{n-r}{t-k}}{\binom{n}{t}}.$$

636 The  $O(1/\sqrt{n})$  bound now follows directly from Lemma E.2 and Lemma E.3.  $\square$

637 Next, we prove Lemma 4.3.

638 **Lemma 4.3.** Fix any deterministic algorithm  $\mathcal{A}$ . For any  $t = \Theta_1(n)$ ,  $\mathbb{E}[\text{cost}_{\mathcal{A}}(t)] = \Omega(\sqrt{n})$ .

*Proof.* Let  $\sigma$  be a permutation of  $[t]$  and  $\mathcal{O}_{\sigma}$  the event that  $a_{\sigma(1)} < a_{\sigma(2)} < \dots < a_{\sigma(t)}$ . For any deterministic algorithm  $\mathcal{A}$ , the choice of the position  $\pi(a_t)$  where to place the  $t$ -th element depends only on  $\sigma$ . Let  $k = \sigma^{-1}(t)$  be the relative rank of the  $t$ -th element. Since the distribution of  $k$  is uniform in  $[t]$  (see the proof of Proposition 3.3), then we have that:

$$\Pr\left[\frac{t}{4} \leq k \leq \frac{3t}{4}\right] = \frac{1}{2}$$

Conditioned on that event  $k = \Theta_1(t)$  so we are in the conditions of Lemma 4.2. Therefore, the probability of each rank given the observations is at most  $O(1/\sqrt{n})$ . Therefore, there is a constant  $c$  such that:

$$\Pr\left[|\text{rk}(a_{(t)}) - \pi(a_{(t)})| < c\sqrt{n} \mid \frac{t}{4} \leq k \leq \frac{3t}{4}\right] \leq \frac{1}{2}$$

639 Finally, we observe that:

$$\begin{aligned} \mathbb{E}[\text{cost}_{\mathcal{A}}(t)] &\geq \frac{1}{2} \cdot \mathbb{E}\left[|\text{rk}(a_{(t)}) - \pi(a_{(t)})| \mid \frac{t}{4} \leq k \leq \frac{3t}{4}\right] \\ &\geq \frac{1}{2} \cdot c\sqrt{n} \cdot \Pr\left[|\text{rk}(a_{(t)}) - \pi(a_{(t)})| \geq c\sqrt{n} \mid \frac{t}{4} \leq k \leq \frac{3t}{4}\right] \geq \frac{c\sqrt{n}}{4}. \end{aligned}$$

640  $\square$

641 We are now ready to prove Theorem 2.

642 *Proof of Theorem 2.* For any deterministic algorithm, sum the bound in Lemma 4.3 for  $\Theta(n)$  time  
643 steps. For randomized algorithms, the same bound extends via Yao's minimax principle. The reason  
644 is that a randomized algorithm can be seen as a distribution on deterministic algorithms parametrized  
645 by the random bits it uses. If a randomized algorithm obtains less than  $O(n\sqrt{n})$  inversions in  
646 expectation, then it should be possible to fix the random bits and obtain a deterministic algorithm  
647 with the same performance.  $\square$

## 648 F Hardness of Online Ranking with Adversarial Ordering

649 **Proposition F.1.** If the ordering  $\sigma$  of the arrival of elements is adversarial, then any algorithm has  
650 cost  $\Omega(n^2)$  in expectation.

651 *Proof.* At a high level, we construct an ordering such that at each iteration, the arrived element is  
652 either the largest or smallest element not yet observed with probability  $1/2$  each. Since the algorithm  
653 cannot distinguish between the two cases, it suffers a linear cost in expectation at each arrival.

654 Formally, we define  $\sigma$  inductively. At round  $t$ , let  $i_{t,-}$  and  $i_{t,+}$  be the minimum and maximum  
655 indices of the elements arrived previously. We define  $\sigma(t)$  such that  $\sigma(t) = a_{i_{t,-}+1}$  with probability  
656  $1/2$  and  $\sigma(t) = a_{i_{t,+}-1}$  with probability  $1/2$ . Thus, the  $t$ th element arrived is either the smallest or  
657 largest element not yet arrived.



658 The main observation is that the pairwise comparisons at time  $t$  are identical whether  $a_{(t)} = a_{i_t, -+1}$   
 659 or  $a_{(t)} = a_{i_t, +-1}$ . This is since all the elements previously arrived are either maximal or minimal  
 660 and there is no elements that are between  $a_{i_t, -+1}$  and  $a_{i_t, +-1}$  that have previously arrived. Thus the  
 661 decision of the algorithm is *independent* of the randomization of the adversary for the  $t$ th element.  
 662 Thus for any learned rank at time  $t$ , in expectation over the randomization of the adversary for the  
 663 element arrived at time  $t$ , the learned rank is at expected distance of the true rank at least  $n/4$  for  
 664  $t \leq n/2$ . Thus the total cost is  $\Omega(n^2)$  in expectation.  $\square$