Parsimonious Quantile Regression of Financial Asset Tail Dynamics via Sequential Learning

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Supplementary Material

Appendix A. The Proof of the Existence of HTQF's Unique Probability Distribution

The proof idea is to show that HTQF is continuously differentiable, is strictly monotonically increasing over $(0,1)$, and approaches $-\infty/+\infty$ as τ tends to $0/1$. So the inverse function of HTQF exists and is a cumulative distribution function.

The HTQF has the specification:

$$Q(\tau|\mu,\sigma,u,v) = \mu + \sigma Z_\tau(\frac{e^{uZ_\tau}}{A}+1)(\frac{e^{-vZ_\tau}}{A}+1) = \mu + \sigma g(Z_\tau), \tag{1}$$

where $g(x) = x(\frac{e^{ux}}{A}+1)(\frac{e^{-vx}}{A}+1)$. Z_τ is the quantile function of the standard normal distribution, so we only need to prove that $g(x)$ is continuously differentiable, is strictly monotonically increasing over $(-\infty,+\infty)$, and approaches $-\infty/+\infty$ as x tends to $-\infty/+\infty$. Obviously $g(x)$ is continuously differentiable and $\lim_{x\to-\infty/+\infty} g(x) = -\infty/+\infty$. To prove the monotonicity, we calculate the derivative of $g(x)$:

$$g'(x) = (\frac{e^{ux}}{A}+1)(\frac{e^{-vx}}{A}+1) + xu\frac{e^{ux}}{A}(\frac{e^{-vx}}{A}+1) - xv(\frac{e^{ux}}{A}+1)\frac{e^{-vx}}{A} \tag{2}$$

$$= \frac{e^{(u-v)x}}{A^2}(1+(u-v)x) + \frac{e^{ux}}{A}(1+ux) + \frac{e^{-vx}}{A}(1-vx) + 1 \tag{3}$$

$$= \frac{1}{A^2}h((u-v)x) + \frac{1}{A}h(ux) + \frac{1}{A}h(-vx) + 1. \qquad (h(x) = e^x(1+x)) \tag{4}$$

Next we prove $h(x) \geq -1$, $\forall x$. This is equivalent to $1+x \geq -e^{-x}$, or $1+x+e^{-x} \geq 0$, $\forall x$. A simple monotonic analysis on the function $1+x+e^{-x}$ can reveal that its global minimum is reached at $x=0$, so $1+x+e^{-x} \geq 2 \geq 0$. So, $h(x) \geq -1$ and

$$g'(x) \geq -\frac{1}{A^2} - \frac{1}{A} - \frac{1}{A} + 1. \tag{5}$$

If we choose $A \geq 3$, then $g'(x) \geq -\frac{1}{9} - \frac{1}{3} - \frac{1}{3} + 1 = \frac{2}{9} > 0$ holds for all x. So $g(x)$ is strictly monotonically increasing and our proof is completed.

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Appendix B. A Brief Introduction to LSTM

In machine learning, LSTM is a type of recurrent neural network model designed for capturing both long-term and short-term dependencies or complex dynamics in sequential data. Recently remarkable successes have been made in its applications like speech recognition, machine translation, protein structure prediction, etc. Mathematically, LSTM is a highly composite nonlinear parametric function that maps a sequence of vectors $x_1, \dots, x_n$ to another sequence of vectors $y_1, \dots, y_n$ (or to just one vector y), through hidden state vectors $h_1, \dots, h_n$. Examples include machine translation from a Chinese sentence to an English sentence, and the classification of a music clip to its genre.

Before describing the full mathematics of it, we first introduce the simple recurrent neural network model, which is the understructure of LSTM and has the functional form:

$$h_j = \sigma_h(W_h x_j + U_h h_{j-1} + b_h), \tag{6}$$

$$y_j = \sigma_y(W_y h_j + b_y). \tag{7}$$

W_h, U_h, b_h, W_y, b_y are the parameters and σ_h, σ_y are nonlinear activation functions. It models a nonlinear functional relationship between $x_1, \dots, x_n$ and $y_1, \dots, y_n$. One can stack this structure multiple times to get multi-layered or hierarchical recurrent neural network, which is a type of deep learning model.

LSTM extends this structure and has the equations:

$$f_j = \sigma_g(W_f x_j + U_f h_{j-1} + b_f), \tag{8}$$

$$i_j = \sigma_g(W_i x_j + U_i h_{j-1} + b_i), \tag{9}$$

$$o_j = \sigma_g(W_o x_j + U_o h_{j-1} + b_o), \tag{10}$$

$$g_j = \sigma_h(W_g x_j + U_g h_{j-1} + b_g), \tag{11}$$

$$c_j = f_j * c_{j-1} + i_j * g_j, \tag{12}$$

$$h_j = o_j * \sigma_h(c_j), \tag{13}$$

and at last, the output y_j is a nonlinear function of h_j, like:

$$y_j = \sigma_h(W_y h_j + b_y). \tag{14}$$

All W, U, b are parameters and σ_g, σ_h are nonlinear activation functions, which are chosen as logistic function and tanh function respectively in this paper. $*$ represents element-wise multiplication of two vectors. In the case of outputting only one vector y, one can use the average of all hidden state vectors $\frac{1}{n}\sum h_j$ or just the last one h_n, e.g., $y = \sigma_h(W_y h_n + b_y)$, like the Equation (8) in our paper.

The tanh function and logistic function are the two most popular activation functions in neural network models, playing a role of nonlinear transformation with finite range. Multiple compositions of these activation functions can approximate complex nonlinear relationships between input vectors $x_1, \dots, x_n$ and output vectors $y_1, \dots, y_n$ or y.

Appendix C. Statistical Testing of VaR Forecasts

Consider a sequence of realized returns or observations $\{r_{t'}\}$ and a sequence of VaR or quantile forecasts $\{q^{t'}\}$ of a fixed probability level τ by any model. In order to implement the testing procedure, we need the definition of hitting sequence of quantile violations:

$$I_{t'} = \begin{cases} 1 & \text{if } r_{t'} < q^{t'} \\ 0 & \text{if } r_{t'} \geq q^{t'} \end{cases} . \tag{15}$$

Ideally, $\{I_{t'}\}$ should be an i.i.d. Bernoulli distribution sequence with parameter τ. To test that, Kupiec's unconditional coverage test [3] checks if the unconditional distribution of $\{I_{t'}\}$ is the Bernoulli distribution, i.e., if the proportion of quantile violations is equal to τ. Christoffersen's independence test [1] checks if $I_{t'}$ is independent of $I_{t'-1}$, i.e., current violation (or not) is independent of previous violation (or not). The mixed conditional coverage test jointly checks these two null hypotheses. One can refer to [2] for details of the three tests. We report the statistics of these tests for $\tau = 0.01, 0.05, 0.1$ VaR forecasts in the following tables. In some sense, the smaller the test statistic, the better the forecasts. The results show that our LSTM-HTQF model does quite well in some cases.

Table 1: Unconditional coverage test for $\tau = 0.01$ quantile forecasts. The test statistic shown below has a chi-square distribution with one degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 3.8415. * represents the threshold is exceeded. In the parentheses, we report the number of quantile violations given by each model against the ideal number of violations.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	14.4440* (35/17)	4.6253* (15/8)	2.0955 (12/8)	21.5459* (33/13)	2.1763 (12/8)	6.2953* (17/9)
GARCH-t	3.9938* (26/17)	4.6253* (15/8)	1.0776 (5/8)	3.1830 (20/13)	0.6900 (10/8)	1.8839 (13/9)
EGARCH-t	5.8339* (28/17)	8.8965* (18/8)	0.0627 (7/8)	6.1990* (23/13)	**0.2423** (9/8)	3.8175 (15/9)
GJR-GARCH-t	3.9938* (26/17)	4.6253* (15/8)	1.0776 (5/8)	3.1830 (20/13)	0.6900 (10/8)	1.8839 (13/9)
AR-EGARCH-t	6.8642* (29/17)	8.8965* (18/8)	0.0627 (7/8)	7.3869* (24/13)	**0.2423** (9/8)	6.2953* (17/9)
AR-GJR-GARCH-t	6.8642* (29/17)	5.9239* (16/8)	1.0776 (5/8)	6.1990* (23/13)	0.6900 (10/8)	2.7780 (14/9)
LSTM-TQR	9.1330* (31/17)	**0.0036** (8/8)	**0.0133** (8/8)	11.4468* (27/13)	2.0919 (4/8)	**0.0553** (8/9)
LSTM-HTQF	4.8762* (27/17)	0.1779 (7/8)	2.1592 (4/8)	**1.6731** (18/13)	0.3710 (6/8)	3.1876 (4/9)

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	1.9494 (16/11)	**0.0395** (13/12)	**0.1439** (11/12)	5.1291* (21/12)	0.3603 (10/12)	4.6935* (15/8)	2.6422 (13/8)	1.7514 (12/8)
GARCH-t	0.1080 (10/11)	1.6210 (17/12)	1.7326 (8/12)	**1.0269** (16/12)	0.6968 (15/12)	0.1665 (7/8)	0.4629 (10/8)	**0.1214** (9/8)
EGARCH-t	0.1080 (10/11)	0.9862 (9/12)	4.0185* (6/12)	2.7314 (7/12)	Inf (0/12)	1.4109 (5/8)	0.4629 (10/8)	1.0174 (11/8)
GJR-GARCH-t	0.1080 (10/11)	1.6210 (17/12)	1.7326 (8/12)	**1.0269** (16/12)	0.6968 (15/12)	0.1665 (7/8)	0.4629 (10/8)	**0.1214** (9/8)
AR-EGARCH-t	0.1080 (10/11)	0.9862 (9/12)	5.6421* (5/12)	2.7314 (7/12)	Inf (0/12)	1.4109 (5/8)	**0.1188** (9/8)	0.4679 (10/8)
AR-GJR-t	0.1080 (10/11)	1.6210 (17/12)	0.4641 (10/12)	**1.0269** (16/12)	3.4917 (19/12)	0.1665 (7/8)	0.4629 (10/8)	0.1318 (7/8)
LSTM-TQR	0.0768 (12/11)	0.1439 (11/12)	5.6421* (5/12)	4.0185* (6/12)	**0.0803** (13/12)	0.4048 (10/8)	1.7413 (12/8)	2.6548 (13/8)
LSTM-HTQF	**0.0004** (11/11)	0.9862 (9/12)	2.7314 (7/12)	2.7314 (7/12)	0.0883 (11/12)	**0.0021** (8/8)	1.0098 (11/8)	3.7148 (14/8)

Table 2: Independence test for $\tau = 0.01$ quantile forecasts. The test statistic shown below has a chi-square distribution with one degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 3.8415. * represents the threshold is exceeded.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	4.3485*	4.9017*	0.3815	1.3335	1.8207	0.9185
GARCH-t	12.3925*	4.9017*	0.0656	1.1056	0.2667	1.7643
EGARCH-t	6.6850*	7.7473*	0.1289	0.7068	0.2157	1.2932
GJR-GARCH-t	12.3925*	4.9017*	0.0656	1.1056	0.2667	1.7643
AR-EGARCH-t	6.3010*	7.7473*	0.1289	0.5980	0.2157	0.9185
AR-GJR-GARCH-t	10.6726*	4.4169*	0.0656	0.7068	0.2667	1.5153
LSTM-TQR	5.5872*	**3.4722**	0.1686	1.1391	**0.0423**	3.5864
LSTM-HTQF	**3.2329**	4.0133*	**0.0419**	**0.5027**	4.5182*	**0.0371**

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	0.4697	2.3621	0.1987	0.8081	**0.1681**	0.5647	0.4295	0.3660
GARCH-t	**0.1825**	1.4367	0.1048	1.6354	0.3798	0.1217	0.2532	0.2051
EGARCH-t	**0.1825**	3.7741	0.0589	0.0802	Inf	**0.0620**	0.2532	0.3071
GJR-GARCH-t	**0.1825**	1.4367	0.1048	1.6354	0.3798	0.1217	0.2532	0.2051
AR-EGARCH-t	**0.1825**	3.7741	**0.0408**	0.0802	Inf	**0.0620**	**0.2048**	0.2535
AR-GJR-t	**0.1825**	1.4367	0.1641	1.6354	0.6114	0.1217	0.2532	**0.1237**
LSTM-TQR	0.2633	0.1987	**0.0408**	**0.0589**	0.2848	0.2494	0.3655	0.4300
LSTM-HTQF	0.2210	**0.1328**	0.0802	0.0802	0.2035	0.1592	0.3067	0.4994

Table 3: Conditional coverage test for $\tau = 0.01$ quantile forecasts. The test statistic shown below has a chi-square distribution with two degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 5.9915. * represents the threshold is exceeded.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	18.7924*	9.5270*	2.4770	22.8794*	3.9970	7.2138*
GARCH-t	16.3863*	9.5270*	1.1432	4.2886	0.9567	3.6482
EGARCH-t	12.5188*	16.6438*	0.1916	6.9058*	**0.4580**	5.1107
GJR-GARCH-t	16.3863*	9.5270*	1.1432	4.2886	0.9567	3.6482
AR-EGARCH-t	13.1652*	16.6438*	0.1916	7.9849*	**0.4580**	7.2138*
AR-GJR-GARCH-t	17.5368*	10.3408*	1.1432	6.9058*	0.9567	4.2934
LSTM-TQR	14.7202*	**3.4758**	**0.1819**	12.5860*	2.1342	3.6416
LSTM-HTQF	8.1091*	4.1912	2.2011	**2.1758**	4.8892	**3.2247**

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	2.4192	2.4016	**0.3426**	5.9372	0.5284	5.2581	3.0717	2.1174
GARCH-t	0.2905	3.0578	1.8375	**2.6623**	1.0766	0.2882	0.7161	0.3264
EGARCH-t	0.2905	4.7603	4.0773	2.8116	Inf	1.4729	0.7161	1.3245
GJR-GARCH-t	0.2905	3.0578	1.8375	**2.6623**	1.0766	0.2882	0.7161	0.3264
AR-EGARCH-t	0.2905	4.7603	5.6829	2.8116	Inf	1.4729	**0.3237**	0.7214
AR-GJR-t	0.2905	3.0578	0.6281	**2.6623**	4.1031	0.2882	0.7161	**0.2556**
LSTM-TQR	0.3401	**0.3426**	5.6829	4.0773	0.3650	0.6542	2.1068	3.0849
LSTM-HTQF	**0.2215**	1.1190	2.8116	2.8116	**0.2918**	**0.1613**	1.3165	4.2142

Table 4: Unconditional coverage test for $\tau = 0.05$ quantile forecasts. The test statistic shown below has a chi-square distribution with one degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 3.8415. * represents the threshold is exceeded. In the parentheses, we report the number of quantile violations given by each model against the ideal number of violations.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	**0.0011** (86/86)	0.3939 (37/41)	**0.0044** (38/38)	2.4133 (78/65)	**0.0001** (38/38)	**0.0039** (43/43)
GARCH-t	0.3385 (91/86)	0.0187 (40/41)	0.0544 (37/38)	5.6727* (85/65)	0.9345 (44/38)	0.1609 (46/43)
EGARCH-t	1.2564 (96/86)	0.4302 (45/41)	0.0544 (37/38)	7.4529* (88/65)	0.6517 (43/38)	3.5414 (56/43)
GJR-GARCH-t	0.3385 (91/86)	0.0187 (40/41)	0.0544 (37/38)	5.6727* (85/65)	0.9345 (44/38)	0.1609 (46/43)
AR-EGARCH-t	2.3894 (100/86)	0.9315 (47/41)	0.3452 (42/38)	12.4481* (95/65)	5.5387* (53/38)	4.6981* (58/43)
AR-GJR-GARCH-t	1.7796 (98/86)	**0.0006** (41/41)	0.0693 (40/38)	8.7620* (90/65)	2.5384 (48/38)	1.6916 (52/43)
LSTM-TQR	3.4629 (103/86)	0.0338 (42/41)	0.3452 (42/38)	1.4192 (75/65)	4.4979* (26/38)	2.0944 (53/43)
LSTM-HTQF	2.0738 (99/86)	0.2141 (38/41)	0.0544 (37/38)	**1.1440** (74/65)	0.1183 (36/38)	0.0087 (44/43)

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	0.7963 (49/55)	1.6265 (52/62)	5.7942* (44/62)	4.4883* (46/62)	9.7465* (38/60)	2.6174 (31/41)	0.8898 (46/40)	**0.4084** (44/40)
GARCH-t	0.2483 (59/55)	2.1412 (73/62)	0.2136 (58/62)	2.9242 (75/62)	5.7512* (79/60)	1.2097 (34/41)	**0.6203** (45/40)	0.9056 (46/40)
EGARCH-t	0.2176 (52/55)	**0.2136** (58/62)	2.8660 (49/62)	0.0388 (60/62)	50.2420* (15/60)	2.6174 (31/41)	1.9715 (49/40)	2.9405 (51/40)
GJR-GARCH-t	0.2483 (59/55)	2.1412 (73/62)	0.2136 (58/62)	2.9242 (75/62)	5.7512* (79/60)	1.2097 (34/41)	**0.6203** (45/40)	0.9056 (46/40)
AR-EGARCH-t	0.2176 (52/55)	0.7487 (55/62)	3.3631 (48/62)	**0.0043** (61/62)	50.2420* (15/60)	2.6174 (31/41)	0.8898 (46/40)	2.4465 (50/40)
AR-GJR-t	0.2483 (59/55)	1.1861 (70/62)	**0.1084** (59/62)	3.8208 (77/62)	8.9937* (84/60)	0.3552 (37/41)	**0.6203** (45/40)	**0.4084** (44/40)
LSTM-TQR	**0.0349** (54/55)	0.7487 (55/62)	3.3631 (48/62)	3.9036* (47/62)	0.9056 (53/60)	**0.2833** (44/41)	1.9715 (49/40)	5.3279* (55/40)
LSTM-HTQF	0.1065 (53/55)	1.7932 (72/62)	3.3631 (48/62)	3.3631 (48/62)	**0.6631** (54/60)	1.2097 (34/41)	3.4446 (52/40)	2.4465 (50/40)

Table 5: Independence test for $\tau = 0.05$ quantile forecasts. The test statistic shown below has a chi-square distribution with one degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 3.8415. * represents the threshold is exceeded.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	0.6556	2.6259	0.5439	0.0907	9.6087*	1.4745
GARCH-t	4.8553*	1.8534	**0.4467**	0.0460	8.9506*	4.2899*
EGARCH-t	**0.5051**	2.2816	3.7523	0.1508	6.7614*	1.5349
GJR-GARCH-t	4.8553*	1.8534	**0.4467**	0.0460	8.9506*	4.2899*
AR-EGARCH-t	0.8202	1.7986	1.0207	0.0047	6.6385*	2.4007
AR-GJR-GARCH-t	3.2234	1.6303	0.7649	**0.0031**	6.7665*	1.1088
LSTM-TQR	0.5932	**0.3261**	4.8689*	1.6276	**1.1215**	2.1964
LSTM-HTQF	3.0207	0.7929	**0.4467**	0.8434	4.8378*	**0.2667**

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	0.3128	**0.0204**	0.1141	0.3807	2.4858	2.4616	5.6162*	5.1314*
GARCH-t	**0.2395**	0.0301	0.0270	3.9111*	0.3417	0.1521	**0.1339**	**0.1934**
EGARCH-t	0.9292	0.2393	0.5287	0.3924	0.3798	**0.0323**	0.3787	6.9599*
GJR-GARCH-t	**0.2395**	0.0301	0.0270	3.9111*	0.3417	0.1521	**0.1339**	**0.1934**
AR-EGARCH-t	0.9292	0.1007	0.6797	0.3170	0.3798	**0.0323**	0.1613	6.6805*
AR-GJR-t	**0.2395**	0.2977	**0.0180**	3.3817	**0.2344**	0.0609	**0.1339**	5.1314*
LSTM-TQR	0.6826	0.1215	3.9029*	0.4506	4.8997*	1.1244	6.3987*	8.1391*
LSTM-HTQF	0.8008	1.5993	0.5261	**0.0089**	1.1647	2.9727	0.7414	6.6805*

Table 6: Conditional coverage test for $\tau = 0.05$ quantile forecasts. The test statistic shown below has a chi-square distribution with two degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 5.9915. * represents the threshold is exceeded.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	**0.6567**	3.0198	0.5483	2.5040	9.6088*	1.4784
GARCH-t	5.1938	1.8722	**0.5010**	5.7187	9.8851*	4.4508
EGARCH-t	1.7615	2.7118	3.8067	7.6037*	7.4131*	5.0764
GJR-GARCH-t	5.1938	1.8722	**0.5010**	5.7187	9.8851*	4.4508
AR-EGARCH-t	3.2096	2.7301	1.3659	12.4529*	12.1772*	7.0988*
AR-GJR-GARCH-t	5.0030	1.6309	0.8342	8.7651*	9.3049*	2.8004
LSTM-TQR	4.0561	**0.3599**	5.2141	3.0468	5.6194	4.2908
LSTM-HTQF	5.0945	1.0070	**0.5010**	**1.9873**	**4.9561**	**0.2754**

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	1.1092	1.6469	5.9083	4.8689	12.2323*	5.0789	6.5060*	5.5398
GARCH-t	**0.4877**	2.1713	0.2406	6.8353*	6.0929*	1.3617	**0.7542**	**1.0990**
EGARCH-t	1.1469	**0.4529**	3.3947	0.4312	50.6218*	2.6497	2.3502	9.9004*
GJR-GARCH-t	**0.4877**	2.1713	0.2406	6.8353*	6.0929*	1.3617	**0.7542**	**1.0990**
AR-EGARCH-t	1.1469	0.8494	4.0428	**0.3213**	50.6218*	2.6497	1.0511	9.1270*
AR-GJR-t	**0.4877**	1.4838	**0.1264**	7.2025*	9.2281*	**0.4161**	**0.7542**	5.5398
LSTM-TQR	0.7175	0.8701	7.2660*	4.3542	5.8053	1.4078	8.3701*	13.4671*
LSTM-HTQF	0.9072	3.3925	3.8892	3.3720	**1.8278**	4.1824	4.1860	9.1270*

Table 7: Unconditional coverage test for $\tau = 0.1$ quantile forecasts. The test statistic shown below has a chi-square distribution with one degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 3.8415. * represents the threshold is exceeded. In the parentheses, we report the number of quantile violations given by each model against the ideal number of violations.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	2.5266 (152/171)	6.3314* (61/82)	2.9169 (63/77)	1.0161 (120/131)	0.5567 (70/76)	**0.1013** (84/87)
GARCH-t	0.0127 (170/171)	5.1218* (63/82)	0.0093 (76/77)	0.7045 (140/131)	0.3441 (81/76)	0.2226 (91/87)
EGARCH-t	0.1361 (176/171)	2.6914 (68/82)	0.3397 (72/77)	0.5609 (139/131)	1.3792 (86/76)	2.4655 (101/87)
GJR-GARCH-t	0.0127 (170/171)	5.1218* (63/82)	0.0093 (76/77)	0.7045 (140/131)	0.3441 (81/76)	0.2226 (91/87)
AR-EGARCH-t	0.7154 (182/171)	**0.4513** (76/82)	0.1464 (80/77)	1.8988 (146/131)	3.0702 (91/76)	4.4403* (106/87)
AR-GJR-GARCH-t	0.2792 (178/171)	3.5642 (66/82)	0.9436 (85/77)	3.0046 (150/131)	0.6773 (83/76)	1.8311 (99/87)
LSTM-TQR	**0.0023** (172/171)	6.3314* (61/82)	0.1147 (74/77)	1.2310 (143/131)	**0.1420** (73/76)	0.3402 (92/87)
LSTM-HTQF	0.4725 (180/171)	4.0498* (65/82)	**0.0006** (77/77)	**0.1484** (135/131)	0.6773 (83/76)	0.1297 (90/87)

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	3.7023 (92/111)	6.0259* (98/123)	14.4691* (85/123)	11.4422* (89/123)	15.7898* (81/120)	9.7119* (56/81)	0.3678 (75/80)	0.6990 (73/80)
GARCH-t	0.2246 (106/111)	10.8205* (159/123)	0.0363 (121/123)	0.8824 (133/123)	8.2363* (151/120)	0.9716 (73/81)	**0.0112** (81/80)	**0.0000** (80/80)
EGARCH-t	0.9701 (101/111)	0.3206 (129/123)	4.2047* (102/123)	0.5897 (115/123)	29.3127* (68/120)	3.3969 (66/81)	0.4727 (86/80)	0.6637 (87/80)
GJR-GARCH-t	0.2246 (106/111)	10.8205* (159/123)	0.0363 (121/123)	0.8824 (133/123)	8.2363* (151/120)	0.9716 (73/81)	**0.0112** (81/80)	**0.0000** (80/80)
AR-EGARCH-t	0.9701 (101/111)	**0.0091** (122/123)	3.4232 (104/123)	0.0360 (125/123)	26.9110* (70/120)	2.1682 (69/81)	0.3272 (85/80)	0.1236 (83/80)
AR-GJR-t	0.3312 (105/111)	4.9291* (147/123)	**0.0000** (123/123)	2.8086 (141/123)	7.7280* (150/120)	0.3916 (76/81)	**0.0112** (81/80)	0.0560 (78/80)
LSTM-TQR	0.1835 (115/111)	0.2286 (118/123)	1.8340 (109/123)	**0.0000** (123/123)	1.6420 (107/120)	**0.0393** (83/81)	0.8416 (88/80)	0.8638 (88/80)
LSTM-HTQF	**0.0737** (108/111)	0.5674 (131/123)	3.8034 (103/123)	0.9260 (113/123)	**0.6196** (112/120)	0.2568 (77/81)	0.1154 (83/80)	0.4893 (86/80)

Table 8: Independence test for $\tau = 0.1$ quantile forecasts. The test statistic shown below has a chi-square distribution with one degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 3.8415. * represents the threshold is exceeded.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	3.3858	0.4903	1.2473	0.1247	3.3491	5.9673*
GARCH-t	6.7318*	**0.2933**	0.9235	**0.0010**	8.4342*	9.6478*
EGARCH-t	3.1115	0.3517	**0.0104**	0.1460	3.2349	5.0258*
GJR-GARCH-t	6.7318*	**0.2933**	0.9235	**0.0010**	8.4342*	9.6478*
AR-EGARCH-t	1.3954	0.5936	0.0630	0.0031	1.0705	5.7199*
AR-GJR-GARCH-t	5.5712*	0.5666	0.0445	0.2619	7.3817*	4.4703*
LSTM-TQR	**0.0021**	1.3428	1.9503	1.0813	**0.6443**	3.1314
LSTM-HTQF	0.0926	1.5988	1.3278	0.3156	5.7355*	**2.5901**

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	0.2696	0.1033	0.1357	0.0364	1.4991	**1.2283**	9.2574*	5.1972*
GARCH-t	0.0031	0.0200	0.4747	2.5164	0.5957	2.8232	3.0373	1.5698
EGARCH-t	2.6627	0.0272	**0.0210**	1.8532	0.2280	1.4321	2.7129	1.7915
GJR-GARCH-t	0.0031	0.0200	0.4747	2.5164	0.5957	2.8232	3.0373	1.5698
AR-EGARCH-t	2.6627	**0.0012**	0.6682	1.6556	0.3534	1.9739	2.4808	**1.0868**
AR-GJR-t	**0.0001**	0.0159	1.3486	2.4694	0.0396	3.5536	3.0373	1.2252
LSTM-TQR	0.9786	1.3083	1.8123	**0.0097**	1.7826	2.0200	3.2069	5.1636*
LSTM-HTQF	0.0352	0.0674	0.0426	0.0428	**0.0153**	2.1133	**1.0026**	1.5995

Table 9: Conditional coverage test for $\tau = 0.1$ quantile forecasts. The test statistic shown below has a chi-square distribution with two degree of freedom. The threshold for rejecting the null hypothesis with 95% confidence level is 5.9915. * represents the threshold is exceeded.

(a)

Method\Stock Index	S&P 500	NASDAQ 100	HSI	Nikkei 225	DAX	FTSE 100
GARCH	5.9125	6.8217*	4.1642	1.1408	3.9058	6.0686*
GARCH-t	6.7445*	5.4151	0.9327	0.7056	8.7783*	9.8704*
EGARCH-t	3.2476	3.0431	0.3501	0.7069	4.6141	7.4913*
GJR-GARCH-t	6.7445*	5.4151	0.9327	0.7056	8.7783*	9.8704*
AR-EGARCH-t	2.1108	**1.0450**	**0.2093**	1.9018	4.1407	10.1602*
AR-GJR-GARCH-t	5.8504	4.1307	0.9881	3.2666	8.0590*	6.3014*
LSTM-TQR	**0.0044**	7.6741*	2.0650	2.3123	**0.7863**	3.4716
LSTM-HTQF	0.5651	5.6486	1.3284	**0.4640**	6.4128*	**2.7198**

(b)

Method\Asset	USDEUR	USDGBP	USDCHF	USDJPY	USDAUD	US2Y	US10Y	US30Y
GARCH	3.9719	6.1292*	14.6048*	11.4786*	17.2889*	10.9402*	9.6252*	5.8962
GARCH-t	0.2276	10.8406*	**0.5110**	3.3988	8.8320*	3.7947	3.0485	1.5698
EGARCH-t	3.6327	0.3478	4.2257	2.4428	29.5407*	4.8290	3.1856	2.4552
GJR-GARCH-t	0.2276	10.8406*	**0.5110**	3.3988	8.8320*	3.7947	3.0485	1.5698
AR-EGARCH-t	3.6327	**0.0103**	4.0914	1.6916	27.2643*	4.1421	2.8080	**1.2104**
AR-GJR-t	0.3313	4.9450	1.3486	5.2779	7.7676*	3.9452	3.0485	1.2812
LSTM-TQR	1.1621	1.5369	3.6462	**0.0097**	3.4247	**2.0592**	4.0485	6.0274*
LSTM-HTQF	**0.1089**	0.6348	3.8460	0.9688	**0.6349**	2.3701	**1.1180**	2.0888

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