
Supplementary material for "Adaptive Active Hypothesis Testing under Limited Information"

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Abstract

1 This supplementary document contains the proofs of propositions, lemmas, and
2 theorems.

3 A Extended proofs

4 A.1 Additional notation

5 In the following proofs we will use some additional notation which we introduce here. Define the
6 random variables

$$Z_s(j_1, j_2) = \log H_s(j_1, j_2), \quad H_s(j_1, j_2) = \frac{f(Y(s), j_1, w(s))}{f(Y(s), j_2, w(s))}$$

7 so as that

$$S_t(j_1, j_2) = \log \frac{\nu_{j_1}(t)}{\nu_{j_2}(t)} = \sum_{s < t} Z_s(j_1, j_2),$$

8 Let $V(j_1, j_2)$ denote the variance of $Z_s(j_1, j_2)$, and

$$V = \max_{j_1, j_2 \in \mathcal{J}} V(j_1, j_2).$$

9 Note that $\Delta_w \leq \Delta^M < 1$, implies that $V < \infty$.

10 The quantity $I(j^*)$ captures the *maximal information* attainable from the system when the incoming
11 job type is j^* ,

$$I(j^*) = \max_{w \in \mathcal{W}} \left\{ \sum_{j \in \mathcal{J}} I(j^*, j, w) \right\}, \quad I(j^*, j, w) = \begin{cases} \Delta_w \log \left(\frac{1 + \Delta_w}{1 - \Delta_w} \right), & j \in \mathcal{J}_{w, -j^*}, \\ 0, & j \in \mathcal{J}_{w, +j^*}. \end{cases}$$

12 The term $\sigma(j^*)$ is the *maximal slack* given that the incoming job type is j^* , and denotes how
13 accurately α_w approximates q_{w, j^*} , specifically

$$\sigma(j^*) = \max_{w \in \mathcal{W}} \{q_{w, j^*} - \alpha_w\}.$$

14 A.2 Proof of Theorem 1

15 As stated in Section 3, the proof is based on showing that

- 16 • If $\mathbb{P}\{\hat{j} \neq j^*\}$ is small, then $\sum_{j \neq j^*} S(j^*, j)$ is large with high probability.

17 • If t is small, then $\sum_{j \neq j^*} S_t(j^*, j)$ is small with high probability.

18 These properties will be shown respectively in Proposition 1(S) and 2(S).

19 **Proposition 1 (S).** *It holds that*

$$\mathbb{P}\left\{\sum_{j \neq j^*} S(j^*, j) < (J-1) \log \frac{1-\delta}{\delta}\right\} \leq \mathbb{P}\{\nu_{j^*}(T^*) \leq \delta\} =: \gamma(\delta).$$

20 *Proof.* Denote the event $B = \left\{\sum_{j \neq j^*} S(j^*, j) < (J-1) \log \frac{1-\delta}{\delta}\right\}$, and we aim to show that

21 $\mathbb{P}\{B|\hat{j} = j^*\} = 0$. It holds that

$$\begin{aligned} \hat{j} = j^* &\Rightarrow \frac{\nu_{j^*}(T^*)}{\nu_j(T^*)} > \frac{1-\delta}{\delta}, \quad \forall j \neq j^* \\ &\Rightarrow S_{j^*,j} > \log \frac{1-\delta}{\delta}, \quad \forall j \neq j^* \\ &\Rightarrow \sum_{j \neq j^*} S_{j^*,j} > (J-1) \log \frac{1-\delta}{\delta} \Rightarrow \bar{B}, \end{aligned}$$

22 and therefore $\mathbb{P}\{B|\hat{j} = j^*\} = 0$. We conclude, since

$$\begin{aligned} \mathbb{P}\{B\} &= \mathbb{P}\{B|\hat{j} \neq j^*\} \mathbb{P}\{\hat{j} \neq j^*\} + \underbrace{\mathbb{P}\{B|\hat{j} = j^*\}}_{=0} \mathbb{P}\{\hat{j} = j^*\} \\ &= \mathbb{P}\{B|\hat{j} \neq j^*\} \mathbb{P}\{\hat{j} \neq j^*\} \leq \mathbb{P}\{\hat{j} \neq j^*\} \leq \mathbb{P}\{\nu_{j^*}(T^*) \leq \delta\}. \end{aligned}$$

23 □

24 Before the proof of Proposition 2(S), observe that the notation in Appendix A.1 are required.

25 **Proposition 2 (S).** *Given $\epsilon > 0$, it holds that*

$$\mathbb{P}\left\{\max_{t \leq T} \sum_{j \neq j^*} S_t(j^*, j) \geq TK(\epsilon)\right\} \leq \frac{(J-1)^2 V}{T\epsilon^2},$$

26 for every $T > 0$, where

$$K(\epsilon) = I(j^*) + 2\sigma(j^*)(J-1) \log \left(\frac{1+\Delta^M}{1-\Delta^M}\right) + \epsilon.$$

27 *Proof.* Let us rewrite $S_t(j^*, j)$ as follows

$$\begin{aligned} S_t(j^*, j) &= \sum_{s \leq t} (Z_s(j^*, j) - \mathbb{E}[Z_s(j^*, j)]) \\ &\quad + \sum_{s \leq t} (\mathbb{E}[Z_s(j^*, j)] - I(j^*, j, w(s))) + \sum_{s \leq t} I(j^*, j, w(s)) \\ &= M_{1,t}(j^*, j) + M_{2,t}(j^*, j) + M_{3,t}(j^*, j), \end{aligned}$$

28 and we analyze these three terms separately.

29 The last term, by definition of $I(j^*)$ is such that

$$\sum_{j \neq j^*} M_{3,t}(j^*, j) = \sum_{s \leq t} \sum_{j \neq j^*} I(j^*, j, w(s)) \leq tI(j^*) \leq TI(j^*).$$

30 The second term is such that

$$\begin{aligned} M_{2,t}(j^*, j) &= \sum_{s \leq t} (\mathbb{E}[Z_s(j^*, j)] - I(j^*, j, w(s))) \\ &\leq \sum_{s \leq t} (2q_{w(s),j^*} - 1 - \Delta_{w(s)}) \log \frac{1+\Delta_{w(s)}}{1-\Delta_{w(s)}} \\ &\leq 2t\sigma(j^*) \log \frac{1+\Delta^M}{1-\Delta^M} \leq 2T\sigma(j^*) \log \frac{1+\Delta^M}{1-\Delta^M} \end{aligned}$$

31 hence

$$\sum_{j \neq j^*} M_{2,t}(j^*, j) \leq T(J-1)2\sigma(j^*) \log \frac{1 + \Delta^M}{1 - \Delta^M}$$

32 It then holds that

$$\sum_{j \neq j^*} M_{1,t}(j^*, j) + M_{2,t}(j^*, j) + M_{3,t}(j^*, j) \geq TK(\epsilon)$$

33 implies that $\sum_{j \neq j^*} M_{1,t}(j^*, j) \geq T\epsilon$, and therefore

$$\mathbb{P}\left\{\max_{t \leq T} \sum_{j \neq j^*} S_t(j^*, j) \geq TK(\epsilon)\right\} \leq \mathbb{P}\left\{\max_{t \leq T} \sum_{j \neq j^*} M_{1,t}(j^*, j) \geq T\epsilon\right\}$$

34 Now, observe that $\sum_{j \neq j^*} M_{1,t}(j^*, j)$ is a \mathcal{L}^2 -martingale. Hence, we can apply Doob's inequality to
35 obtain

$$\mathbb{P}\left\{\max_{t \leq T} \sum_{j \neq j^*} M_{1,t}(j^*, j) \geq T\epsilon\right\} \leq \frac{H}{T^2\epsilon^2}$$

36 where

$$H = \mathbb{E}\left[\left(\sum_{t \leq T} \sum_{j \neq j^*} (z_s(j^*, j) - \mathbb{E}[z_s(j^*, j)])\right)^2\right] \leq T(J-1)^2V.$$

37 Hence, we conclude that

$$\mathbb{P}\left\{\max_{t \leq T} \sum_{j \neq j^*} S_t(j^*, j) \geq TK(\epsilon)\right\} \leq \frac{(J-1)^2V}{T\epsilon^2}.$$

38 □

39 *Proof.* (of Theorem 1.) Fix $\epsilon > 0$ and define

$$t_\delta = \frac{(J-1)}{K(\epsilon)} \log \frac{1-\delta}{\delta}$$

40 where $K(\epsilon)$ is defined in Proposition 2(S). Due to the law of total probability, it holds that

$$\begin{aligned} \mathbb{P}\{T^* \leq t_\delta\} &\leq \mathbb{P}\left\{T^* \leq t_\delta, \sum_{j \neq j^*} S(j^*, j) \geq (J-1) \log \frac{1-\delta}{\delta}\right\} \\ &\quad + \mathbb{P}\left\{\sum_{j \neq j^*} S(j^*, j) < (J-1) \log \frac{1-\delta}{\delta}\right\}. \end{aligned}$$

41 By means of Proposition 2(S), the first term can be bounded as follows

$$\mathbb{P}\left\{T^* \leq t_\delta, \sum_{j \neq j^*} S(j^*, j) \geq (J-1) \log \frac{1-\delta}{\delta}\right\} \leq \mathbb{P}\left\{\max_{t \leq t_\delta} \sum_{j \neq j^*} S_t(j^*, j) \geq t_\delta K(\epsilon)\right\} \leq \frac{(J-1)^2V}{t_\delta \epsilon^2},$$

42 where the first inequality follows from the definition of t_δ . Further, the second term can be bounded
43 via Proposition 1(S)

$$\mathbb{P}\left\{\sum_{j \neq j^*} S(j^*, j) < \log(J-1) \frac{1-\delta}{\delta}\right\} \leq \mathbb{P}\{\nu_{j^*}(T^*) \leq \delta\} = \gamma(\delta).$$

44 These bounds together yield

$$\mathbb{P}\{T^* \leq t_\delta\} \leq \kappa(\delta), \quad \kappa(\delta) := \gamma(\delta) + \frac{(J-1)^2V}{t_\delta \epsilon^2},$$

45 and we conclude by observing that

$$\begin{aligned}\mathbb{E}[T^*] &\geq \mathbb{E}[T^* | T^* > t_\delta] \mathbb{P}\{T^* > t_\delta\} \\ &\geq t_\delta(1 - \mathbb{P}\{T^* \leq t_\delta\}) \geq t_\delta(1 - \kappa(\delta)).\end{aligned}$$

46 Note that since $\lim_{\delta \rightarrow 0} \gamma(\delta) = 0$ and $\lim_{\delta \rightarrow 0} t_\delta = \infty$, it holds that

$$\lim_{\delta \rightarrow 0} \kappa(\delta) = 0.$$

47 Therefore there exists $\bar{\delta} > 0$ such that for every $\delta < \bar{\delta}$, it holds that $\kappa(\delta) < 1/2$. Hence, for every
48 $\delta < \bar{\delta}$, it holds that

$$\mathbb{E}[T^*] \geq \frac{1}{2}t_\delta = \frac{(J-1)}{2K(\epsilon)} \log \frac{1-\delta}{\delta} \geq \frac{1}{2(\Delta_M + 2\sigma(j^*)) \log\left(\frac{1+\Delta^M}{1-\Delta^M}\right) + \frac{2\epsilon}{J-1}} \log \frac{1-\delta}{\delta}$$

49 since

$$K(\epsilon) \leq (J-1)(\Delta_M + 2\sigma(j^*)) \log\left(\frac{1+\Delta^M}{1-\Delta^M}\right) + \epsilon.$$

50

□

51 A.3 Control of the belief vector evolution

52 We now control the ratio between coordinates of the belief vector under the IB policy. Specifically, at
53 a certain time t , we bound the probability that $\nu_j(t) > \nu_{j^*}(t)$, and investigate how this probability
54 evolves with t .

55 The first proposition presents the bound and is based on a coupling argument.

56 **Proposition 3 (S).** *Under the IB update policy, for every $j \neq j^*$, it holds that*

$$\mathbb{P}\left\{\frac{\nu_{j^*}(t)}{\nu_j(t)} \leq \epsilon\right\} \leq \epsilon, \quad \forall \epsilon > 0, t > 0.$$

57 *A Bayesian coupled system.* We first introduce an alternative way to describe the IB update rule. At
58 time t , sample a value $U(t)$ from a uniform random variable in $[0, 1]$. Assume to have chosen the
59 worker-class $w(t) \in \mathcal{W}$, then

$$\nu_j(t+1) = \frac{f(y(t), j, w(t))\nu_j(t)}{\sum_{i \in \mathcal{J}} f(y(t), i, w(t))\nu_i(t)},$$

60 where

$$f(y(t), j, w(t)) = \begin{cases} \alpha_{w(t)}, & y(t) = g_{w(t),j}, \\ 1 - \alpha_{w(t)}, & y(t) = -g_{w(t),j}, \end{cases} \quad y(t) = \begin{cases} g_{w(t),j^*}, & U(t) < q_{w(t),j^*}, \\ -g_{w(t),j^*}, & U(t) \geq q_{w(t),j^*}. \end{cases}$$

61 We now introduce a coupled belief-process $\mu(t)$, which evolves in parallel with $\nu(t)$ according to the
62 following rule

$$\mu_j(t+i) = \frac{f(y^p(t), j, w(t))\mu_j}{\sum_{i \in \mathcal{J}} f(y^p(t), i, w(t))\mu_i}, \quad (1)$$

63 where

$$y^p(t) = \begin{cases} y(t), & U(t) < \alpha_{w(t)}, \\ -y(t), & U(t) \in [\alpha_{w(t)}, q_{w(t),j^*}), \\ y(t), & U(t) \geq q_{w(t),j^*}. \end{cases} = \begin{cases} g_{w(t),j^*}, & U(t) < \alpha_{w(t)}, \\ -g_{w(t),j^*}, & U(t) \geq \alpha_{w(t)}. \end{cases}$$

64 The peculiarity of the $\mu(t)$ belief vector is that $f(y^p(t), j, w(t))$ is the probability of having response
65 $y^p(t)$ in the *pessimistic* system given where $q_{w,j} = \alpha_w$ for every $w \in \mathcal{W}$ and $j \in \mathcal{J}$. Hence, $\mu(t)$ is
66 updated according to the Bayesian update rule and therefore it represents the *real* posterior probability
67 vector in the pessimistic *fictitious* scenario.

68 This parallel process is introduced due to the following Lemma.

69 **Lemma 1 (S).** *If $\nu(0) = \mu(0)$, then*

$$\frac{\nu_{j^*}(t)}{\nu_j(t)} \geq \frac{\mu_{j^*}(t)}{\mu_j(t)}, \quad \forall t \geq 0.$$

70 *Proof.* Observe that if $w(t) \in \mathcal{W}_{j^*,j}$

$$\frac{\nu_{j^*}(t+1)}{\nu_j(t+1)} = \frac{\nu_{j^*}(t)}{\nu_j(t)} \times \begin{cases} \frac{1+\Delta_{w(t)}}{1-\Delta_{w(t)}}, & \text{if } U(t) \leq q_{w(t),j^*} \\ \frac{1-\Delta_{w(t)}}{1+\Delta_{w(t)}}, & \text{if } U(t) > q_{w(t),j^*} \end{cases}$$

71 and

$$\frac{\mu_{j^*}(t+1)}{\mu_j(t+1)} = \frac{\mu_{j^*}(t)}{\mu_j(t)} \times \begin{cases} \frac{1+\Delta_{w(t)}}{1-\Delta_{w(t)}}, & \text{if } U(t) \leq \alpha_{w(t)} \\ \frac{1-\Delta_{w(t)}}{1+\Delta_{w(t)}}, & \text{if } U(t) > \alpha_{w(t)}. \end{cases}$$

72 on the other hand, if $w(t) \notin \mathcal{W}_{j^*,j}$, it holds that

$$\frac{\nu_{j^*}(t+1)}{\nu_j(t+1)} = \frac{\nu_{j^*}(t)}{\nu_j(t)}, \quad \frac{\mu_{j^*}(t+1)}{\mu_j(t+1)} = \frac{\mu_{j^*}(t)}{\mu_j(t)}.$$

73 These relations conclude the proof together with the initial condition $\nu(0) = \mu(0)$. \square

74 *Proof of Proposition 3(S).* From Lemma 1(S) it follows immediately that

$$\mathbb{P}\left\{\frac{\nu_{j^*}(t)}{\nu_j(t)} \leq \beta\right\} \leq \mathbb{P}\left\{\frac{\mu_{j^*}(t)}{\mu_j(t)} \leq \beta\right\},$$

75 and define $B_{t,j^*,j}$ the event

$$B_{t,j^*,j} = \left\{\frac{\mu_{j^*}(t)}{\mu_j(t)} \leq \beta\right\}.$$

76 Observe that, over $B_{t,j^*,j}$, it holds that

$$\prod_{s < t} f(y^p(s), j^*, w(s)) \leq \beta \prod_{s < t} f(y^p(s), j, w(s)).$$

77 Hence,

$$\begin{aligned} \mathbb{P}\{B_{t,j^*,j} | j^*\} &= \int_{B_{t,j}} \prod_{s < t} f(y^p(s), j^*, w(s)) d\left((y^p(1), w(1)), \dots, (y^p(t-1), w(t-1))\right) \\ &\leq \beta \int_{B_{t,j}} \prod_{s < t} f(y^p(s), j, w(s)) d\left((y^p(1), w(1)), \dots, (y^p(t-1), w(t-1))\right) \\ &= \beta \mathbb{P}\{B_{t,j^*,j} | j\} \leq \beta. \end{aligned} \quad \square$$

78 This result indicates how likely it is that we are on the wrong path, i.e., $\nu_{j^*}(t)$ should not be lower
79 than $\nu_j(t)$. The next proposition gives us a more explicit bound, however it depends on the sequence
80 of actions chosen up to time t .

81 **Proposition 4 (S).** *Under the IB update policy, for every $j \neq j^*$, it holds that*

$$\mathbb{P}\left\{\frac{\nu_{j^*}(t)}{\nu_j(t)} < M\right\} \leq (1+M)(1-\Delta_m^2)^{|\mathcal{W}_{j^*,j}(t)|/2}, \quad \forall M > 0. \quad (2)$$

82 where

$$|\mathcal{W}_{j^*,j}(t)| = |\{s < t, w(s) \in \mathcal{W}_{j^*,j}\}|.$$

83 *Proof.* According to the definitions in Appendix A.1, it holds that

$$\mathbb{P}\left\{\frac{\nu_{j^*}(t)}{\nu_j(t)} \leq M\right\} = \mathbb{P}\{S_t(j^*, j) \leq \log M\}.$$

84 For $\gamma \in [-1, 0]$, it holds that

$$\begin{aligned} \mathbb{P}\{S_t(j^*, j) \leq \log M\} &= \mathbb{P}\{S_t(j^*, j) \leq 0\} + \mathbb{P}\{S_t(j^*, j) \in (0, \log M]\} \\ &\leq \frac{\mathbb{E}[e^{\gamma S_t(j^*, j)}]}{\mathbb{E}[e^{\gamma S_t(j^*, j)} | S_t(j^*, j) \leq 0]} + \frac{\mathbb{E}[e^{\gamma S_t(j^*, j)}]}{\mathbb{E}[e^{\gamma S_t(j^*, j)} | S_t(j^*, j) \in (0, \log M)]} \\ &\leq \mathbb{E}[e^{\gamma S_t(j^*, j)}] + \frac{\mathbb{E}[e^{\gamma S_t(j^*, j)}]}{e^{-\log M}} = \mathbb{E}[e^{\gamma S_t(j^*, j)}](1 + M). \end{aligned}$$

85 We now consider $\tilde{H}_s(\gamma; j^*, j) = \mathbb{E}[(H_s(j^*, j))^\gamma]$. In case $w(s) \notin \mathcal{W}_{j^*, j}$, it holds that $H_s(j^*, j) = 1$
 86 with probability 1, and therefore $\tilde{H}_s(\gamma; j^*, j) = 1$ for every $\gamma \in [-1, 0]$. On the other hand, consider
 87 the case where $w(s) = w \in \mathcal{W}_{j^*, j}$. Then, for $\gamma = -1$, it holds that

$$\tilde{H}_s(-1; j^*, j) = q_{w, j^*} \left(\frac{\alpha_w}{1 - \alpha_w} \right)^{-1} + (1 - q_{w, j^*}) \left(\frac{1 - \alpha_w}{\alpha_w} \right)^{-1}$$

88 which is lower or equal than 1 since $q_{w, j^*} \geq \alpha_w > 1/2$. Moreover, note that $\tilde{H}_s(0; j^*, j) = 1$ and
 89 since $\tilde{H}_s(\cdot; j^*, j)$ is a convex function, it holds that $\tilde{H}_s(-1/2; j^*, j) < 1$. Hence, if $w(s) \in \mathcal{W}_{j^*, j}$,
 90 it holds

$$\begin{aligned} \mathbb{E}[(H_s(j^*, j))^{-1/2}] &= q_{w, j^*} \sqrt{\frac{1 - \alpha_w}{\alpha_w}} + (1 - q_{w, j^*}) \sqrt{\frac{\alpha_w}{1 - \alpha_w}} \\ &\leq \alpha_w \sqrt{\frac{1 - \alpha_w}{\alpha_w}} + (1 - \alpha_w) \sqrt{\frac{\alpha_w}{1 - \alpha_w}} = 2\sqrt{\alpha_w} \sqrt{1 - \alpha_w}. \end{aligned}$$

91 Finally, observe that

$$2\sqrt{\alpha_w} \sqrt{1 - \alpha_w} = \sqrt{1 - \Delta_w^2} \leq \sqrt{1 - \Delta_m^2}$$

92 The proof is concluded by observing that

$$\begin{aligned} \frac{1}{1 + M} \mathbb{P}\left\{ \frac{\nu_{j^*}(t)}{\nu_j(t)} \leq M \right\} &\leq \mathbb{E}[e^{-S_t(j^*)/2}] = \prod_{s < t} \mathbb{E}[(H_s(j^*, j))^{-1/2}] \\ &= \prod_{s < t} \tilde{H}_s(1/2; j^*, j) = \prod_{s < t, w(s) \in \mathcal{W}_{j^*, j}} \tilde{H}_s(1/2; j^*, j) \\ &\leq (1 - \Delta_m^2)^{|\mathcal{W}_{j^*, j}(t)|/2}. \end{aligned}$$

93 □

94 The argument in the above proof is similar to [1, Lemma 1]. The important difference is that in [1]
 95 every action is able to distinguish hypotheses j^* and j , and therefore the exponent on the right-hand-
 96 side of (2) is t in that case, instead of $|\mathcal{W}_{j^*, j}(t)|$. Our model only satisfies Assumption 2; however if
 97 a given action selection policy continues exploring each pairs of hypotheses, we deduce the following
 98 corollary.

99 **Corollary 1.** *It holds that*

$$\lim_{t \rightarrow \infty} |\mathcal{W}_{j^*, j}(t)| = \infty \quad \Rightarrow \quad \lim_{t \rightarrow \infty} \mathbb{P}\left\{ \frac{\nu_{j^*}(t)}{\nu_j(t)} < M \right\} = 0 \quad \forall M > 0.$$

100 As a consequence of Corollary 1, we deduce that any candidate policy for action selection shouldn't
 101 have the property where there exists $\bar{t} > 0$ and $\bar{j} \in \mathcal{J}$ such that $w(s) \notin \mathcal{W}_{\bar{j}, j^*}$ for all $s \geq \bar{t}$. Indeed,
 102 in such a case that policy would not be able to completely distinguish \bar{j} from the true hypothesis j^* .

103 A.4 Proof of Theorem 2

104 Before the actual proof of Theorem 2 we need to show that, by using the IBAG algorithm, the decision
 105 maker is expected to obtain a positive amount of information at each step.

106 **Proposition 5 (S).** Under the IBAG, there exists $K > 0$ such that

$$\mathbb{E}[U_{t+1} - U_t] \geq K,$$

107 where

$$U_t = \log \frac{\nu_{j^*}(t)}{1 - \nu_{j^*}(t)}.$$

108 *Proof.* Denote by $w^D = w^D(t)$ the class of workers chosen by the IBAG algorithm at step t , and by
109 $\nu_{j^*} = \nu_{j^*}(t)$. Observe that

$$\begin{aligned} & \mathbb{E}[U_{t+1} - U_t] \\ &= \mathbb{E}\left[\log \frac{f(Y_t, j^*, w^D)}{\sum_{j \neq j^*} \frac{f(Y_t, j, w^D) \nu_j(t)}{\sum_{j \neq j^*} \nu_j(t)}}\right] \\ &= q_{w^D, j^*} \log \frac{\alpha_{w^D}}{\alpha_{w^D} - \frac{\Delta_{w^D} \nu_{w^D, -j^*}}{1 - \nu_{j^*}}} + (1 - q_{w^D, j^*}) \log \frac{1 - \alpha_{w^D}}{1 - \alpha_{w^D} + \frac{\Delta_{w^D} \nu_{w^D, -j^*}}{1 - \nu_{j^*}}} \\ &\geq \alpha_{w^D} \log \frac{\alpha_{w^D}}{\alpha_{w^D} - \frac{\Delta_{w^D} \nu_{w^D, -j^*}}{1 - \nu_{j^*}}} + (1 - \alpha_{w^D}) \log \frac{1 - \alpha_{w^D}}{1 - \alpha_{w^D} + \frac{\Delta_{w^D} \nu_{w^D, -j^*}}{1 - \nu_{j^*}}} \\ 110 &= D_{KL}\left(\left(\frac{1 + \Delta_{w^D}}{2}, \frac{1 - \Delta_{w^D}}{2}\right) \parallel \left(\frac{1 + \Delta_{w^D}}{2} - \frac{\Delta_{w^D} \nu_{w^D, -j^*}}{1 - \nu_{j^*}}, \frac{1 - \Delta_{w^D}}{2} + \frac{\Delta_{w^D} \nu_{w^D, -j^*}}{1 - \nu_{j^*}}\right)\right) \\ &\geq D_{KL}\left(\left(\frac{1 + \Delta^m}{2}, \frac{1 - \Delta^m}{2}\right) \parallel \left(\frac{1 + \Delta^m}{2} - \Delta^m \frac{\nu_{w^D, -j^*}}{1 - \nu_{j^*}}, \frac{1 - \Delta^m}{2} + \Delta^m \frac{\nu_{w^D, -j^*}}{1 - \nu_{j^*}}\right)\right) \\ &=: K(\nu_{w^D, -j^*}, \nu_{j^*}) \geq 0 \end{aligned}$$

111 where $D_{KL}(\cdot \parallel \cdot)$ denotes the Kullback-Leibler divergence. Observe that

$$\frac{\partial K(x, y)}{\partial x} \geq 0, \quad \frac{\partial K(x, y)}{\partial y} \geq 0,$$

112 and we aim to bound $K(x, y)$ away from zero with high probability. We fix $\tilde{\nu} < 1/J$ and distinguish
113 two possible cases:

114

115 *Case 1:* The decision maker is in an explorative phase, i.e., it does not have a clear feeling about
116 which type the incoming job is. In this phase, there exists $\bar{w} \in \mathcal{W}$ such that

$$\nu_{-\bar{w}} \geq \tilde{\nu}.$$

117 This yields that

$$\nu_{j^*} \geq 0, \quad \nu_{w^D, -j^*} \geq \nu_{-w^D} \geq \tilde{b} \nu_{-\bar{w}} \geq \tilde{b} \tilde{\nu},$$

118 where \tilde{b} is defined in Lemma 3(S) which is proved in Appendix B. Therefore

$$K(\nu_{w^D, -j^*}, \nu_{j^*}) \geq K(\tilde{b} \tilde{\nu}, 0) > 0.$$

119 *Case 2:* The decision maker is in an exploitative phase, i.e., it does have a clear feeling about which
120 type the incoming job is. In this phase, for every $w \in \mathcal{W}$, it holds that

$$\nu_{-w} < \tilde{\nu}.$$

121 The following lemma states that indeed in this case there is a job-type which is clearly the most likely
122 type of the incoming job. The proof is provided in Appendix B.

123 **Lemma 2 (S).** If $\nu_{-w} < \frac{1}{J}$ for every $w \in \mathcal{W}$, then there exists $\bar{j} \in \mathcal{J}$ such that

$$\bigcap \mathcal{J}_{+w} = \{\bar{j}\}, \quad \mathcal{J}_{+w} = \begin{cases} \mathcal{J}_w, & \text{if } \sum_{j \in \mathcal{J}_w} \nu_j \geq \sum_{j \notin \mathcal{J}_w} \nu_j, \\ \mathcal{J} \setminus \mathcal{J}_w, & \text{if } \sum_{j \in \mathcal{J}_w} \nu_j < \sum_{j \notin \mathcal{J}_w} \nu_j. \end{cases}$$

124 At this point, we distinguish two subcases:

125 2a) Assume $\bar{j} = j^*$. It means that we are on the correct path towards the end of the learning
126 process, and

$$\nu_{w^D, -j^*} = \nu_{-w^D}, \quad \nu_{j^*} \geq 1 - \frac{J-1}{\tilde{b}} \nu_{-w^D},$$

127 where the second relation holds due to Lemma 3(S) and

$$1 = \nu_{j^*} + \sum_{j \neq j^*} \nu_j \leq \nu_{j^*} + \sum_{j \neq j^*} \nu_{-w(j)} \leq \nu_{j^*} + (J-1) \frac{\nu_{-w^D}}{\tilde{b}}.$$

128 Hence,

$$K(\nu_{w^D, -j^*}, \nu_{j^*}(t)) \geq K(\nu_{-w^D}, 1 - \frac{J-1}{\tilde{b}} \nu_{-w^D}) = K(1, 1 - \frac{J-1}{\tilde{b}}) > 0.$$

129 2b) Assume $\bar{j} \neq j^*$. It means that we are on the wrong path towards the end of the learning
130 process, and we would like to show that this is unlikely to happen. Denote by $w(j)$ a class
131 of workers belonging to $\mathcal{W}_{j, \bar{j}}$, it holds that

$$\nu_{j^*}(t) < \nu_{-w(j^*)} \leq \tilde{\nu}, \quad \nu_{\bar{j}}(t) \geq 1 - (J-1)\tilde{\nu},$$

132 since

$$1 = \nu_{\bar{j}} + \sum_{j \neq \bar{j}} \nu_j \leq \nu_{\bar{j}} + \sum_{j \neq \bar{j}} \nu_{-w(j)} \leq \nu_{\bar{j}} + (J-1)\tilde{\nu}.$$

133 Therefore

$$\frac{\nu_{j^*}(t)}{\nu_{\bar{j}}(t)} \leq \frac{\tilde{\nu}}{1 - (J-1)\tilde{\nu}}$$

134 and Proposition 1 yields that

$$\mathbb{P}\left\{\frac{\nu_{j^*}(t)}{\nu_{\bar{j}}(t)} \leq \frac{\tilde{\nu}}{1 - (J-1)\tilde{\nu}}\right\} \leq \frac{\tilde{\nu}}{1 - (J-1)\tilde{\nu}}$$

135 Hence in this second phase, we obtain that

$$\begin{aligned} \mathbb{E}[U_{t+1} - U_t] &= \mathbb{E}[U_{t+1} - U_t | \bar{j} = j^*] \mathbb{P}\{\bar{j} = j^*\} + \mathbb{E}[U_{t+1} - U_t | \bar{j} \neq j^*] \mathbb{P}\{\bar{j} \neq j^*\} \\ &\geq \mathbb{E}[U_{t+1} - U_t | \bar{j} = j^*] \mathbb{P}\{\bar{j} = j^*\} \\ &\geq \mathbb{E}[U_{t+1} - U_t | \bar{j} = j^*] (1 - \mathbb{P}\left\{\frac{\nu_{j^*}(t)}{\nu_{\bar{j}}(t)} \leq \frac{\tilde{\nu}}{1 - (J-1)\tilde{\nu}}\right\}) \\ &\geq K\left(1, 1 - \frac{J-1}{\tilde{b}}\right) \frac{1 - J\tilde{\nu}}{1 - J\tilde{\nu} + \tilde{\nu}}. \end{aligned}$$

136 Define the following events

$$A_{\tilde{\nu}}(t) = \{\exists \bar{w} : \nu_{-\bar{w}}(t) \geq \tilde{\nu}\}, \quad B_{\tilde{\nu}}(t) = A_{\tilde{\nu}}(t)^C.$$

137 We just showed that

$$\mathbb{E}[U_{t+1} - U_t | A_{\tilde{\nu}}(t)] \geq K(\tilde{b}\tilde{\nu}, 0) =: K_A(\tilde{\nu})$$

138 and that

$$\mathbb{E}[U_{t+1} - U_t | B_{\tilde{\nu}}(t)] \geq K\left(1, 1 - \frac{J-1}{\tilde{b}}\right) \frac{1 - J\tilde{\nu}}{1 - J\tilde{\nu} + \tilde{\nu}} =: K_B(\tilde{\nu}).$$

139 Observing that

$$\begin{aligned} \mathbb{E}[U_{t+1} - U_t] &= \mathbb{E}[U_{t+1} - U_t | A_{\tilde{\nu}}(t)] \mathbb{P}(A_{\tilde{\nu}}(t)) + \mathbb{E}[U_{t+1} - U_t | B_{\tilde{\nu}}(t)] \mathbb{P}(B_{\tilde{\nu}}(t)) \\ &\geq K_A(\tilde{\nu}) \mathbb{P}(A_{\tilde{\nu}}(t)) + K_B(\tilde{\nu}) \mathbb{P}(B_{\tilde{\nu}}(t)) \\ &\geq \min\{K_A(\tilde{\nu}), K_B(\tilde{\nu})\}. \end{aligned}$$

140 Define

$$K_{\tilde{\nu}} = \min\{K_A(\tilde{\nu}), K_B(\tilde{\nu})\} > 0,$$

141 so as that, for $\tilde{\nu} = \frac{1}{J^2}$, it holds that

$$K_{\frac{1}{J^2}} = \min\{K(\frac{\tilde{b}}{J^2}, 0), K(1, 1 - \frac{J-1}{\tilde{b}}) \frac{J^2 - J}{J^2 - J + 1}\}.$$

142 Note that as J grows large both $K(\frac{\tilde{b}}{J^2}, 0)$ and $K(1, 1 - \frac{J-1}{\tilde{b}})$ converges to zero as $\log(\frac{J}{1+J})$. \square

143 *Proof.* (of Theorem 2.) Observe that by the definition of T^* , it holds that $T^* \leq T(j^*)$ where

$$T(j^*) = \inf_t \{\nu_{j^*}(t) > 1 - \delta | j^*\}.$$

144 Hence,

$$\mathbb{E}[T^*] \leq \mathbb{E}[T(j^*)] = \sum_t \mathbb{P}\{T(j^*) > t\}.$$

145 Observe that

$$\nu_{j^*} \leq 1 - \delta \quad \Rightarrow \quad \frac{\nu_{j^*}}{1 - \nu_{j^*}} \leq \frac{1 - \delta}{\delta} = \frac{1}{\delta} - 1 < \frac{1}{\delta},$$

146 and therefore

$$\{T(j^*) > t\} \Rightarrow B(t), \quad B(t) = \{U_t \leq -\log \delta\},$$

147 where

$$U_t = \log \frac{\nu_{j^*}(t)}{1 - \nu_{j^*}(t)}.$$

148 Note that

$$\mathbb{P}\{B(t)\} = \mathbb{P}\{U_t \leq -\log \delta\} = \mathbb{P}\{U_t - \mathbb{E}[U_t] \leq -\log \delta - \mathbb{E}[U_t]\},$$

149 and from Proposition 5(S) we obtain

$$\mathbb{P}\{B(t)\} \leq \mathbb{P}\{U_t - \mathbb{E}[U_t] \leq -\log \delta - U_0 - tK\}.$$

150 At this point, consider any $t \geq \bar{t}(\delta) := \frac{2}{K}\eta(\delta)$, with $K = K_{\frac{1}{J^2}}$ and $\eta(\delta) = -\log \delta - U_0 > 0$ then

$$\begin{aligned} \mathbb{P}\{B(t)\} &\leq \mathbb{P}\{-U_t + \mathbb{E}[U_t] \geq tK - \eta(\delta)\} \\ &\leq \mathbb{P}\{|U_t - \mathbb{E}[U_t]| \geq tK - \eta(\delta)\} \\ &\leq e^{-\frac{(tK - \eta(\delta))^2}{2Ht}} \end{aligned}$$

151 where

$$H = 2 \frac{1 + \Delta^M}{1 - \Delta^M} \geq \max_{s < t} |U_s - U_{s-1} - \mathbb{E}[U_s - U_{s-1}]|.$$

152 and the last inequality is due to Azuma's inequality.

153 Hence,

$$\begin{aligned} \mathbb{E}[T(j^*)] &\leq \bar{t}(\delta) + \sum_{t \geq \bar{t}} \mathbb{P}\{T(j^*) > t\} \\ &\leq \bar{t}(\delta) + \sum_{t \geq \bar{t}(\delta)} e^{-\frac{(tK - \eta(\delta))^2}{2Ht}} \\ &= \bar{t}(\delta) + \sum_{t \geq \bar{t}(\delta)} e^{-\frac{tK^2}{2H}} e^{\frac{K\eta(\delta)}{H}} e^{-\frac{\eta(\delta)^2}{2Ht}} \\ &\leq \bar{t}(\delta) + e^{\frac{K\eta(\delta)}{H}} \sum_{t \geq \bar{t}(\delta)} e^{-\frac{tK^2}{2H}} \\ &= \bar{t}(\delta) + e^{\frac{K\eta(\delta)}{H}} \frac{e^{-\frac{\bar{t}(\delta)K^2}{2H}}}{1 - e^{-\frac{K^2}{2H}}} = \bar{t}(\delta) + \frac{1}{1 - e^{-\frac{K^2}{2H}}}, \end{aligned}$$

154 Theorem 2 follows by defining

$$K_1^u = \frac{2}{K}, \quad K_0^u = -\frac{2U_0}{K} + \frac{1}{1 - e^{-\frac{K^2}{2H}}}.$$

155 For the sake of completeness, note that H is constant in J , and

$$U_0 = -\log(J-1), \quad \frac{1}{K} \sim \frac{1}{\log(\frac{J}{1+J})}.$$

156

□

157 B Proof of the technical lemmas

158 *Proof of Lemma 1.* Given that at time t we observe the belief vector $\boldsymbol{\nu} = \boldsymbol{\nu}(t) \in \mathbb{P}(\mathcal{J})$, and a response
159 is asked to a worker in class $w = w(t)$. For every worker $w \in \mathcal{W}$ and $j \in \mathcal{J}$, define

$$\nu_{w,+j} = \sum_{j \in \mathcal{J}_{w,+j}} \nu_j, \quad \nu_{w,-j} = \sum_{j \in \mathcal{J}_{w,-j}} \nu_j.$$

160 Then if $y = g_{w,j^*}$, i.e., a correct response is observed,

$$\begin{aligned} j \in \mathcal{J}_{w,+j^*} &\Rightarrow \nu_j(t+1) = \frac{\nu_j(1 + \Delta_w)}{\nu_{w,+j^*}(1 + \Delta_w) + \nu_{w,-j^*}(1 - \Delta_w)} \geq \nu_j, \\ j \notin \mathcal{J}_{w,+j^*} &\Rightarrow \nu_j(t+1) = \frac{\nu_j(1 - \Delta_w)}{\nu_{w,+j^*}(1 + \Delta_w) + \nu_{w,-j^*}(1 - \Delta_w)} \leq \nu_j, \end{aligned}$$

161 while, on the other hand, if $y = -g_{w,j^*}$, i.e., a wrong response is observed,

$$\begin{aligned} j \in \mathcal{J}_{w,+j^*} &\Rightarrow \nu_j(t+1) = \frac{\nu_j(1 - \Delta_w)}{\nu_{w,+j^*}(1 - \Delta_w) + \nu_{w,-j^*}(1 + \Delta_w)} \leq \nu_j, \\ j \notin \mathcal{J}_{w,+j^*} &\Rightarrow \nu_j(t+1) = \frac{\nu_j(1 + \Delta_w)}{\nu_{w,+j^*}(1 - \Delta_w) + \nu_{w,-j^*}(1 + \Delta_w)} \geq \nu_j. \end{aligned}$$

162 The analysis of $\nu_{j^*}(t+1)/\nu_j(t+1)$ in the various cases conclude the proof.

163

164 *Proof of Lemma 4.* Lemma 4 is a consequence of the following stronger result.

165 **Lemma 3 (S).** Consider $x, y \in [0, \frac{1}{2}]$ such that $G(x, \Delta^M) \geq G(y, \Delta^m)$, then

$$x \geq \tilde{b}y, \quad \tilde{b} := \frac{\Delta^m}{\Delta^M}. \quad (3)$$

166 In fact, from the monotonicity properties of the function $G(\cdot, \cdot)$, at time t it holds that

$$G(\nu_{-w^D(t)}, \Delta^M) \geq G(\nu_{-w^D(t)}, \Delta_{w^D(t)}) \geq G(\nu_{-w}, \Delta_w) \geq G(\nu_{-w}, \Delta^m),$$

167 and Lemma 3(S) yields $\nu_{-w^D(t)} \geq \tilde{b}\nu_{-w}$ for every $w \in \mathcal{W}$.

168 It remains to prove Lemma 3(S) whose proof is done via a contradiction argument. Assume that

169 $x < \tilde{b}y$. In this case, it holds that

$$\begin{aligned} G(x, \Delta^M) &< G(\tilde{b}y, \Delta^M) \\ &= \frac{(\Delta^M)^2(\tilde{b}y)^2}{1 - (\Delta^M)^2(1 - 2\tilde{b}y)^2} \\ &= \frac{(\Delta^m)^2y^2}{1 - (\Delta^M)^2(1 - 2\tilde{b}y)^2} \\ &= \frac{(\Delta^m)^2y^2}{1 - (\Delta^m)^2(1 - 2y)^2} = G(y, \Delta^m), \end{aligned}$$

170 which is a contradiction. Note that the last equality follows since

$$\begin{aligned}
& \frac{1}{1 - (\Delta^M)^2(1 - 2\tilde{b}y)^2} \leq \frac{1}{1 - (\Delta^m)^2(1 - 2y)^2} \\
\iff & (\Delta^M)^2(1 - 2\tilde{b}y)^2 \leq (\Delta^m)^2(1 - 2y)^2 \\
\iff & \Delta^M(1 - 2\tilde{b}y) \leq \Delta^m(1 - 2y) \\
\iff & y(\Delta^m - \tilde{b}\Delta^M) \leq \frac{1}{2}(\Delta^m - \Delta^M) \\
\iff & y \leq \frac{1}{2} \frac{(\Delta^m - \Delta^M)}{(\Delta^m - \tilde{b}\Delta^M)}
\end{aligned}$$

171 which is true since $y \leq 1/2$ and

$$\frac{(\Delta^m - \Delta^M)}{(\Delta^m - \tilde{b}\Delta^M)} \geq 1.$$

172 *Proof of Lemma 2(S).* First of all, observe that $|\cap \mathcal{J}_{+w}(t)| \leq 1$. In fact, assume $\{j_1, j_2\} \in \cap \mathcal{J}_{+w}(t)$,
173 then consider $\bar{w} \in \mathcal{W}_{j_1, j_2}$, it yields a contradiction since it is not possible that

$$\{j_1, j_2\} \in \mathcal{J}_{+\bar{w}}(t).$$

174 Now, assume that $\cap \mathcal{J}_{+w}(t) = \emptyset$, then, for every $j \in \mathcal{J}$ it is possible to identify $w(j) \in \mathcal{W}$ such that
175 $j \in \mathcal{J}_{-w}(t)$. For this reason, it holds that

$$1 = \sum_{j \in \mathcal{J}} \nu_j(t) \leq \sum_{j \in \mathcal{J}} \nu_{-w(j)}(t) < J \frac{1}{J} = 1.$$

176 Hence, there exists $\bar{j} \in \cap \mathcal{J}_{+w}(t)$.

177

178 C On the effect of the slack

179 In this section we investigate how different the choices of the IBAG algorithm would be if, instead
180 of α_w , we had at our disposal the exact skill parameters $q_{w,j}$. So as to gain better insights, in this
181 section we assume $q_{w,j}$ to be independent from j , i.e.,

$$q_{w,j} = q_w \in [\alpha_w, \alpha_w + \sigma_w],$$

182 and we aim to capture the effect of σ_w on the algorithm decision. Define

$$\Delta_{q(w)} = 2q_w - 1 \in [\Delta_w, \bar{\Delta}_w], \quad \Delta_w = 2\alpha_w - 1, \quad \bar{\Delta}_w = \Delta_w + 2\sigma_w.$$

183 With the knowledge of q_w , the Incomplete Bayesian updating rule described in Section 3 coincides
184 exactly with the classical Bayesian updating rule presented in Section 3 as well. In particular

$$\mathbb{E}[\nu_{j^*}(t+1) | \boldsymbol{\nu}, w(t) = w] - \nu_{j^*}(t) = 4\nu_{j^*}(t) \frac{\Delta_{q(w)}^2 \nu_{w, -j^*}(t)^2}{1 - \Delta_{q(w)}^2 (1 - \nu_{w, -j^*}(t))^2}$$

185 and denote by $w^B(t)$ the class of workers picked by IBAG in this case, i.e.,

$$w^B(t) = \arg \max_{w \in \mathcal{W}} \{G(\nu_{-w}(t), \Delta_{q(w)})\}, \quad G(v, d) = \frac{d^2 v^2}{1 - d^2 (1 - 2v)^2}.$$

186 We recall that the IBAG algorithm, which only knows α_w , at time t picks the class of workers $w^D(t)$
187 maximizing

$$w^D(t) = \arg \max_{w \in \mathcal{W}} \{G(\nu_{-w}(t), \Delta_w)\},$$

188 and we would like to better understand under which condition the choices made in the two cases are
189 different, i.e.,

$$w^B(t) \neq w^D(t),$$

190 and, in case they are different, what is the impact of the error on the performance, i.e.,

$$\text{Err}(\boldsymbol{\nu}) = \mathbb{E}[\nu_{j^*}(t+1) | \boldsymbol{\nu} = \boldsymbol{\nu}(t), w(t) = w^B(t)] - \mathbb{E}[\nu_{j^*}(t+1) | \boldsymbol{\nu} = \boldsymbol{\nu}(t), w(t) = w^D(t)].$$

191 Note that, it follows from the definition of $w^B(t)$ that $\text{Err}(\boldsymbol{\nu}) \geq 0$.

192 **Lemma 4 (S).** *If for every $w \neq w^D(t)$ it holds that*

$$G(\nu_{-w^D(t)}(t), \Delta_{w^D(t)}) \geq G(\nu_{-w}(t), \bar{\Delta}_w)$$

193 *then $w^D(t) = w^B(t)$.*

194 *Proof.* Recall that $G(v, d)$ is increasing in d , hence

$$G(\nu_{-w^D(t)}(t), \Delta_{q(w^D(t))}) \geq G(\nu_{-w^D(t)}(t), \Delta_{w^D(t)}),$$

195 and

$$G(\nu_{-w}(t), \bar{\Delta}_w) \geq G(\nu_{-w}(t), \Delta_{q(w)}), \quad \forall w \in \mathcal{W}.$$

196 Hence, for every $w \neq w^D(t)$, the hypothesis of the lemma holds. \square

197 In particular, this lemma states that if a class of worker is *much more convenient* than the others when
 198 only the lower bound α_w are known, then the same holds when the probabilities q_w are known. Here
 199 we investigate what kind of threshold determines a worker to be much more convenient than the
 200 others. For every $w \in \mathcal{W}$ define the following quantities

$$\begin{aligned} r_w &= G(\nu_{-w^D(t)}(t), \Delta_{w^D(t)}) - G(\nu_{-w}(t), \Delta_w), \\ s_w &= G(\nu_{-w}(t), \bar{\Delta}_w) - G(\nu_{-w}(t), \Delta_w). \end{aligned}$$

201 Note that r_w represents by how much $w^D(t)$ is more convenient than w given that only α_w is known,
 202 in particular

$$G(\nu_{-w^D(t)}(t), \Delta_{w^D(t)}) \geq G(\nu_{-w}(t), \bar{\Delta}_w) \iff r_w \geq s_w.$$

203 **Proposition 6 (S).** *It holds that*

$$\lim_{\sigma_w \rightarrow 0} s_w = 0, \quad \lim_{\nu_{-w}(t) \rightarrow 0} s_w = 0.$$

204 *Proof.* Note that

$$\begin{aligned} s_w &= G(\nu_{-w}(t), \bar{\Delta}_w) - G(\nu_{-w}(t), \Delta_w) \\ &= 4 \frac{\nu_{-w}(t)^2 \sigma_w (\Delta_w + \sigma_w)}{(1 - (\Delta_w + 2\sigma_w)^2 (1 - 2\nu_{-w}(t))^2) (1 - \Delta_w^2 (1 - 2\nu_{-w}(t))^2)} \\ &\leq 4 \frac{\nu_{-w}(t)^2 \sigma_w (\Delta_w + \sigma_w)}{(1 - (2\sigma_w + \Delta_w)^2 (1 - 2\nu_{-w}(t))^2)^2}. \end{aligned}$$

205 The proposition follows by taking the limits. \square

206 As a consequence of this proposition, it follows that for σ_w or $\nu_{-w}(t)$ sufficiently low, it holds that
 207 $r_w \geq s_w$. Recall that when this happens for each $w \neq w^D(t)$, it follows that $w^D(t) = w^B(t)$.

208 Nevertheless, even when the worker chosen is not the optimal one, the error incurred is not large in
 209 many circumstances. The following proposition bounds $\text{Err}(\nu)$ with a linear function of $\max_w \{s_w\}$,
 210 whose dependence on σ_w has been pointed out in the proof of Proposition 6(S).

211 **Proposition 7 (S).** *It holds that*

$$\text{Err}(\nu) \leq 4\nu_{j^*}(t) s_{w^B(t)} \leq 4\nu_{j^*}(t) \max_w \{s_w\}.$$

212 *Proof.* Observe that due to the definition of G and $\text{Err}(\nu)$, we only need to to upper bound the
 213 following difference

$$G(\nu_{-w}(t), \Delta_{q(w)}) - G(\nu_{-w^D(t)}(t), \Delta_{q(w^D)})$$

214 under the constraint that

$$G(\nu_{-w}(t), \Delta_w) \leq G(\nu_{-w^D(t)}(t), \Delta_{w^D}).$$

215 Observe that

$$\begin{aligned} &G(\nu_{-w}(t), \Delta_{q(w)}) - G(\nu_{-w^D(t)}(t), \Delta_{q(w^D)}) \\ &\leq G(\nu_{-w}(t), \bar{\Delta}_w) - G(\nu_{-w^D(t)}(t), \Delta_w) \\ &\leq G(\nu_{-w}(t), \bar{\Delta}_w) - G(\nu_{-w}(t), \Delta_w) = s_w. \end{aligned}$$

216 \square