
Stochastic Network Design in Bidirected Trees

— Supplementary Material —

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Lemma 2. *If condition (7) holds, $z(\pi) - \hat{z}(\pi) \leq \epsilon z(\pi)$ for an arbitrary policy π .*

Proof. To prove the lemma, we first use induction to prove the following statement:

$$z_u(\pi) - \hat{z}_u(\pi) \leq \epsilon z_u(\pi) \quad \forall u \in V$$

For the base case where u is a leaf vertex, we have $z_u(\pi) = h_u^2$ and therefore the error $z_u(\pi) - \hat{z}_u(\pi)$ is bounded by $K_u^z = \frac{\epsilon}{3} z_u(\pi)$.

For the induction step, assume that the statement holds for the two children v and w of u . To prove that it also holds for u , we consider the error introduced by **each term** in recurrence (6) that uses the rounded values $\hat{\nu}_v(\pi)$ and $\hat{\mu}_w(\pi)$ instead of the true values $\nu_v(\pi)$ and $\mu_w(\pi)$. For the term $\hat{\mu}_v(\pi) p_{v \rightsquigarrow w | \pi} \hat{\nu}_w(\pi)$, the introduced error is $(\mu_v(\pi) p_{v \rightsquigarrow w | \pi} \nu_w(\pi) - \hat{\mu}_v(\pi) p_{v \rightsquigarrow w | \pi} \hat{\nu}_w(\pi))$. Using Lemma 1, the error is

$$\leq \left(\frac{\epsilon}{3} \mu_v(\pi) \hat{\nu}_w(\pi) + \frac{\epsilon}{3} \hat{\mu}_v(\pi) \nu_w(\pi) + \frac{\epsilon^2}{3^2} \mu_v(\pi) \nu_w(\pi) \right) p_{v \rightsquigarrow w | \pi} \leq \epsilon \mu_v(\pi) p_{v \rightsquigarrow w | \pi} \nu_w(\pi)$$

where the last inequality holds because the rounded value always underestimates the true value.

Similarly, the error for the term $\hat{\mu}_w(\pi) p_{w \rightsquigarrow v | \pi} \hat{\nu}_v(\pi)$ is bounded by $\epsilon \mu_w(\pi) p_{w \rightsquigarrow v | \pi} \nu_v(\pi)$. For the term $h_u \hat{\mu}_u^{sum}$, the error is $h_u (\mu_u(\pi) - \hat{\mu}_u^{sum}(\pi))$. By using Lemma 1, the error is bounded by $h_u \frac{\epsilon}{3} (\mu_u(\pi) - h_u)$. Similarly, the error for the term $h_u \hat{\nu}_u^{sum}$ is bounded by $h_u \frac{\epsilon}{3} (\nu_u(\pi) - h_u)$. In addition, by the inductive assumption, the errors for $\hat{z}_v(\pi)$ and $\hat{z}_w(\pi)$ are bounded by $\epsilon z_v(\pi)$ and $\epsilon z_w(\pi)$ respectively.

Therefore, the total error for the enumerator of Equation (6) is bounded by

$$\epsilon (z_v(\pi) + z_w(\pi) + \mu_v(\pi) p_{v \rightsquigarrow w | \pi} \nu_w(\pi) + \mu_w(\pi) p_{w \rightsquigarrow v | \pi} \nu_v(\pi) + h_u \nu_{\cdot, u}(\pi) + h_u \mu_u(\pi) - 2 \cdot h_u^2)$$

According to the definition of $\hat{z}_u(\pi)$ in Equation (6), the enumerator is divided by K_u^z , rounded and then multiplied by K_u^z . These operations introduce an additional error K_u^z , which is bounded by ϵh_u^2 based on condition (7) in the paper. By adding the error to the total error above, the error $z_u(\pi) - \hat{z}_u(\pi)$ is bounded by $\epsilon z_u(\pi)$ where the expression of $z_u(\pi)$ is shown in Equation (4) in the paper. Hence, the statement holds for u .

Finally, as defined in the paper, $z(\pi) = z_{root}(\pi)$ and $\hat{z}(\pi) = \hat{z}_{root}(\pi)$. Therefore, the lemma holds. \square

Derivation of the last inequality used to prove Theorem 3.

$$\max_{0 \leq k \leq (n-1)} \frac{c}{\epsilon^6} (k^4 (n-k-1)^4 + k^8 + (n-k-1)^8) \leq c \frac{n^8}{\epsilon^6}$$

Proof. First, we show for any integer $k \in [0, n - 1]$ that

$$\begin{aligned} & k^4(n - k - 1)^4 + k^8 + (n - k - 1)^8 \\ & \leq 2k^4(n - k - 1)^4 + k^8 + (n - k - 1)^8 \\ & = (k^4 + (n - k - 1)^4)^2 \\ & \leq (k^4 + 2k^2(n - k - 1)^2 + (n - k - 1)^4)^2 = \left((k^2 + (n - k - 1)^2)^2 \right)^2 \\ & \leq \left((k^2 + 2k(n - k - 1) + (n - k - 1)^2)^4 \right)^4 = \left((k + n - k - 1)^2 \right)^4 \\ & = (n - 1)^8 \leq n^8 \end{aligned}$$

Therefore, we have

$$\max_{0 \leq k \leq (n-1)} k^4(n - k - 1)^4 + k^8 + (n - k - 1)^8 \leq n^8$$

which proves the fact. □