

5 Supplementary material

5.1 Seriation lemmas

Here, we prove some of the technical lemmas from Section 2.

Lemma 5.1 *Suppose $A \in \mathbf{S}_n$ is a $\{0, 1\}$ R-matrix, and ΠA is a P-matrix, then $\Pi A \Pi^T$ is an R-matrix.*

Proof. Without loss of generality, we can assume that the graph of A is irreducible (otherwise, we simply repeat the proof on each block). If $A \in \mathbf{S}_n$ is an irreducible $\{0, 1\}$ R-matrix, then $\text{diag}(A) = \mathbf{1}$. Let Π be a permutation such that ΠA is a P-matrix, so $C = \Pi A \Pi^T$ is a symmetric P-matrix. Let $1 \leq j < i \leq n$, and suppose $C_{ij} = 1$, then $C_{(i-1)j} = 1$ because $C_{jj} = 1$ and C is a P-matrix. Similarly, because C is symmetric, if $C_{ij} = 1$ then $C_{ji} = 1$ and $C_{(j+1)i} = 1$ because $C_{ii} = 1$ and C is a P-matrix, so $C_{i(j+1)} = 1$. This means that C is an R-matrix. ■

Lemma 5.2 *Suppose $A \in \mathbf{S}_n$ is a $\{0, 1\}$ pre-R matrix, then $\Pi A \Pi^T$ is an R-matrix if and only if $\Pi A^2 \Pi^T$ is an R-matrix.*

Proof. If $A \in \mathbf{S}_n$ is a $\{0, 1\}$ pre-R matrix, then it must be pre-P (cf. remarks above). [15, Th. 6.3] shows that ΠA is a P-matrix iff $\Pi A^2 \Pi^T$ is an R-matrix. Combining this with Lemma 5.1 yields the desired result. ■

Note that a $\{0, 1\}$ R-matrix is also a (symmetric) P-matrix. Note also that Lemma 5.1 shows that if A is pre-R, then $\Pi A \Pi^T$ is an R-matrix, hence a P-matrix, and so is ΠA (it is obtained by permuting the columns of a P-matrix), so A is also pre-P.

Lemma 5.3 *Let $A \in \mathbf{S}_n$, $y \in \mathbb{R}^n$ and suppose we switch the values of y_j and y_{j+1} calling the new vector z , we have*

$$f(y) - f(z) = 4 \sum_{\substack{i=1 \\ i \neq j, i \neq j+1}}^n \left(\frac{y_j + y_{j+1}}{2} - y_i \right) (y_{j+1} - y_j) (A_{ij+1} - A_{ij})$$

Proof. Because A is symmetric, we have

$$\begin{aligned} (f(y) - f(z))/2 &= \sum_{i \neq j, i \neq j+1} A_{ij} (y_i - y_j)^2 + \sum_{i \neq j, i \neq j+1} A_{ij+1} (y_i - y_{j+1})^2 \\ &\quad - \sum_{i \neq j, i \neq j+1} A_{ij} (y_i - y_{j+1})^2 - \sum_{i \neq j, i \neq j+1} A_{ij+1} (y_i - y_j)^2 \\ &= \sum_{i \neq j, i \neq j+1} 2A_{ij} (y_j - y_{j+1}) \left(\frac{y_j + y_{j+1}}{2} - y_i \right) \\ &\quad + \sum_{i \neq j, i \neq j+1} 2A_{ij+1} (y_{j+1} - y_j) \left(\frac{y_j + y_{j+1}}{2} - y_i \right) \end{aligned}$$

which yields the desired result. ■

Lemma 5.4 *Suppose $A = \text{CUT}(u, v)$, and write $w = y_\pi$ the optimal solution to (2). If we call $I = [u, v]$ and I^c its complement, then*

$$w_j \notin [\min(w_I), \max(w_I)], \quad \text{for all } j \in I^c,$$

in other words, the coefficients in w_I and w_{I^c} belong to disjoint intervals.

Proof. Without loss of generality, we can assume that the coefficients of w_I are sorted in increasing order. By contradiction, suppose that there is a w_j such that $j \in I^c$ and $w_j \notin [w_u, w_v]$. Suppose also that w is larger than the mean of coefficients inside I , i.e. $w_j \geq \sum_{i=u+1}^v w_i / (v - u)$. This, combined with our assumption that $w_j \leq w_v$ and Lemma 5.3 means that switching the values of w_j and w_v will decrease the objective by

$$4 \sum_{i=u}^{v-1} \left(\frac{w_j + w_v}{2} - y_i \right) (w_v - w_j)$$

which is positive by our assumptions on w_j and the mean which contradicts optimality. A symmetric result holds if w_j is smaller than the mean. ■

Lemma 5.5 Suppose $A \in \mathbb{R}^{n \times m}$ is a Q -matrix, then $A \circ A^T$ is a conic combination of CUT matrices.

Proof. Suppose, $a \in \mathbb{R}^n$ is a unimodal vector, let us show that the matrix $M = a \circ a^T$ with coefficients $M_{ij} = \min\{a_i, a_j\}$ is a conic combination of CUT matrices. Let $I = \operatorname{argmax}_i a_i$, then I is an index interval $[I_{\min}, I_{\max}]$ because a is unimodal. Call $\bar{a} = \max_i a_i$ and $b = \max_{i \in I^c} a_i$ (with $b = 0$ when $I^c = \emptyset$), the deflated matrix

$$M^- = M - (\bar{a} - b) \operatorname{CUT}(I_{\min}, I_{\max})$$

can be written $M^- = a^- \circ (a^-)^T$ with

$$a^- = a - (\bar{a} - b)v$$

where $v_i = 1$ iff $i \in I$. By construction $|\operatorname{argmax} M^-| > |I|$, i.e. the size of $\operatorname{argmax} M$ increases by at least one, so this deflation procedure ends after at most n iterations. This shows that $a \circ a^T$ is a conic combination of CUT matrices when a is unimodal. Now, we have $(A \circ A^T)_{ij} = \sum_{k=1}^n w_k \min\{A_{ik}, A_{jk}\}$, so $A \circ A^T$ is a sum of n matrices of the form $\min\{A_{ik}, A_{jk}\}$ where each column is unimodal, hence the desired result. ■

5.2 Convex relaxation results

We now prove some of the convex relaxation results obtained in Section 3.

Proposition 5.6 The optimization problem

$$\min_{\{\Pi \in \mathcal{D}_n, e_1^T \Pi v + 1 \leq e_n^T \Pi v\}} \frac{1}{p} \operatorname{Tr}(Y^T \Pi^T L_A \Pi Y) - \frac{\mu}{p} \|P \Pi\|_F^2 \quad (10)$$

is equivalent to problem (5), their objectives differ by a constant. Furthermore, when $\mu \leq \lambda_2(L_A) \lambda_1(Y Y^T)$, this problem is convex.

Proof. Remark that

$$\begin{aligned} \|P \Pi\|_F^2 &= \operatorname{Tr}(\Pi^T P^T P \Pi) = \operatorname{Tr}(\Pi^T P \Pi) \\ &= \operatorname{Tr}(\Pi^T (I - \frac{1}{n} \mathbf{1} \mathbf{1}^T) \Pi) = \operatorname{Tr}(\Pi^T \Pi - \frac{1}{n} \mathbf{1} \mathbf{1}^T) \\ &= \operatorname{Tr}(\Pi^T \Pi) - 1 \end{aligned}$$

where we used the fact that P is the (symmetric) projector matrix onto the orthogonal of $\mathbf{1}$ and Π is doubly stochastic (so $\Pi \mathbf{1} = \Pi^T \mathbf{1} = \mathbf{1}$). We deduce that problem (6) has the same objective function as (5) up to a constant. Moreover, it is convex when $\mu \leq \lambda_2(L_A)$ since the Hessian of the objective is given by

$$\Sigma = \frac{1}{p} Y Y^T \otimes L_A - \frac{\mu}{p} \cdot \mathbf{I} \otimes P$$

and the eigenvalues of $Y Y^T \otimes L_A$, which are equal to $\lambda_i(L_A) \lambda_j(Y Y^T)$ for all i, j in $\{1, \dots, n\}$ are all superior or equal to the eigenvalues of $\mu \cdot \mathbf{I} \otimes P$ which are all smaller than μ . ■

We now show that that minimizing the average of the relaxed problems costs provides in a sense a tighter relaxation to the combinatorial problem 2 than solving individually the relaxed problems.

Proposition 5.7 Let Π_0 be the optimal solution of 2, Π_i^* the optimal solution of 3 with $y = y_i$ and Π_m^* be the optimal solution of 4. Π_0 is an optimal solution to $\min_{\{\Pi \in \mathcal{P}_n, e_1^T \Pi v + 1 \leq e_n^T \Pi v\}} \frac{1}{p} \sum_{i=1}^p y_i^T \Pi^T L_A \Pi y_i$ and

$$\frac{1}{p} \sum_{i=1}^p y_i^T \Pi_i^{*T} L_A \Pi_i^* y_i \leq \frac{1}{p} \sum_{i=1}^p y_i^T \Pi_m^{*T} L_A \Pi_m^* y_i \leq \frac{1}{p} \sum_{i=1}^p y_i^T \Pi_0^T L_A \Pi_0 y_i .$$

Proof. Π_0 is optimal for all problems 2 with $y = y_i$ so it is optimal for $\min_{\{\Pi \in \mathcal{P}_n, e_1^T \Pi v + 1 \leq e_n^T \Pi v\}} \frac{1}{p} \sum_{i=1}^p y_i^T \Pi^T L_A \Pi y_i$. The first inequality comes from the optimality of each Π_i^* for problem 3 with $y = y_i$. The second inequality comes from the optimality of Π_m^* for the relaxed problem 4. ■

With independent constraints (D full rank), at each iteration, the full variable updates in the dual Euclidean projection problem over doubly stochastic matrices are given by

- $Z = \max\{0, x\mathbf{1}^T + \mathbf{1}y^T + Dz g^T - \Pi_0\}$
- $x = \frac{1}{n}(\Pi_0 \mathbf{1} - (y^T \mathbf{1} + 1)\mathbf{1} - Dz g^T \mathbf{1} + Z\mathbf{1})$
- $y = \frac{1}{n}(\Pi_0^T \mathbf{1} - (x^T \mathbf{1} + 1)\mathbf{1} + Z^T \mathbf{1})$
- $z = \frac{1}{\|g\|_2^2} \max\{0, (D^T D)^{-1}(D^T(Z + \Pi_0)g + \delta - D^T x g^T \mathbf{1})\}$.

The convergence of the algorithm can be monitored through the duality gap formed by the difference of the objective of (8) and (9).

5.3 Numerical experiments

	Kendall Sol.	Spectral	QP	QP Reg	QP + 0.1%
Kendall τ	1.00±0.00	0.75±0.00	0.70±0.22	0.73±0.22	0.76±0.16
Spearman ρ	1.00±0.00	0.90±0.00	0.87±0.19	0.88±0.19	0.91±0.16
Comb. Obj.	38520±0	38903±0	42293±14928	41810±13960	43457±23004
# R-constr.	1556±0	1802±0	2029±491	2021±484	2050±747
	QP + 0.2%	QP + 0.5%	QP + 1.1%	QP + 2.4%	QP + 5.1%
Kendall τ	0.79±0.07	0.80±0.04	0.81±0.03	0.83±0.03	0.86±0.02
Spearman ρ	0.93±0.05	0.94±0.03	0.94±0.02	0.96±0.02	0.97±0.01
Comb. Obj.	43227±12475	44970±8456	43748±7989	43064±8105	42575±5779
# R-constr.	2026±485	2116±377	2045±356	2026±358	1978±288
	QP + 10.7%	QP + 22.3%	QP + 47.5%	QP + 100%	
Kendall τ	0.89±0.02	0.93±0.01	0.97±0.01	0.99±0.00	
Spearman ρ	0.98±0.01	0.99±0.00	1.00±0.00	1.00±0.00	
Comb. Obj.	40452±4107	38126±1916	37602±775	37203±125	
# R-constr.	1855±191	1646±110	1545±43	1512±9	

Table 3: Performance metrics (median and stdev over 100 runs of the QP relaxation, for Kendall’s τ , Spearman’s ρ , the objective value in (1), and the number of R-matrix monotonicity constraint violations), comparing Kendall’s original solution with that of the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with an increasing number of pairwise ordering constraints specified, out of the 3422 possible pairs in this problem. Note that the semi-supervised solution actually improves on Kendall’s original solution.

	Kendall Sol.	Spectral	QP	QP Reg	QP + 0.1%
Kendall τ	1.00	0.76	0.86	0.89	0.86
Spearman ρ	1.00	0.90	0.96	0.97	0.97
Comb Obj.	38520.00	38903.00	30862.00	31369.00	32464.00
# R-constr.	1556.00	1802.00	1335.00	1371.00	1465.00
	QP + 0.2%	QP + 0.5%	QP + 1.1%	QP + 2.4%	QP + 5.1%
Kendall τ	0.86	0.87	0.90	0.89	0.90
Spearman ρ	0.96	0.97	0.98	0.98	0.98
Comb Obj.	31082.00	32345.00	32956.00	32209.00	33669.00
# R-constr.	1361.00	1480.00	1514.00	1460.00	1559.00
	QP + 10.7%	QP + 22.3%	QP + 47.5%	QP + 100%	
Kendall τ	0.92	0.96	0.99	1.00	
Spearman ρ	0.99	1.00	1.00	1.00	
Comb Obj.	34303.00	33731.00	35270.00	36758.00	
# R-constr.	1561.00	1456.00	1461.00	1492.00	

Table 4: Performance metrics (best of 100 runs of the QP relaxation, for Kendall’s τ , Spearman’s ρ , the objective value in (1), and the number of R-matrix monotonicity constraint violations), comparing Kendall’s original solution with that of the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with an increasing number of pairwise ordering constraints specified, out of the 3422 possible pairs in this problem. Note that the semi-supervised solution actually improves on Kendall’s original solution.

	No noise	Noise within spectral gap	Large noise
True	1.00±0.00	1.00±0.00	1.00±0.00
Spectral	1.00±0.00	0.96±0.10	0.46±0.29
QP	0.57±0.36	0.52±0.35	0.42±0.32
QP Reg	0.66±0.36	0.56±0.35	0.39±0.32
QP Reg + 0.1%	0.77±0.33	0.38±0.32	0.77±0.33
QP Reg + 0.7%	0.80±0.25	0.78±0.29	0.80±0.27
QP Reg + 1.4%	0.80±0.23	0.78±0.25	0.79±0.20
QP Reg + 2.5%	0.83±0.13	0.83±0.12	0.81±0.10
QP Reg + 4.6%	0.87±0.07	0.86±0.06	0.85±0.09
QP Reg + 8.7%	0.91±0.04	0.90±0.04	0.90±0.04
QP Reg + 16.1%	0.95±0.02	0.95±0.02	0.94±0.02
QP Reg + 29.7%	0.98±0.01	0.98±0.01	0.98±0.01
QP Reg + 54.3%	1.00±0.00	1.00±0.00	1.00±0.00
QP Reg + 100.0%	1.00±0.00	1.00±0.00	1.00±0.00

Table 5: Median \pm stdev. on Spearman’s ρ between the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

	No noise	Noise within spectral gap	Large noise
True	1.00	1.00	1.00
Spectral	1.00	1.00	1.00
QP	1.00	1.00	1.00
QP Reg	1.00	1.00	1.00
QP Reg add 0.1% cons	1.00	0.99	1.00
QP Reg add 0.7% cons	0.97	0.98	0.96
QP Reg add 1.4% cons	0.94	0.95	0.97
QP Reg add 2.5% cons	0.98	0.95	0.95
QP Reg add 4.6% cons	0.95	0.97	0.95
QP Reg add 8.7% cons	0.98	0.99	0.96
QP Reg add 16.1% cons	0.99	0.98	0.99
QP Reg add 29.7% cons	1.00	1.00	1.00
QP Reg add 54.3% cons	1.00	1.00	1.00
QP Reg add 100.0% cons	1.00	1.00	1.00

Table 6: Best Spearman’s ρ between the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

	No noise	Noise within spectral gap	Large noise
True	1.00±0.00	1.00±0.00	1.00±0.00
Spectral	1.00±0.00	0.86±0.14	0.41±0.25
QP	0.49±0.34	0.55±0.31	0.45±0.27
QP Reg	0.50±0.34	0.58±0.31	0.45±0.27
QP Reg + 0.1%	0.65±0.29	0.40±0.26	0.60±0.27
QP Reg + 0.7%	0.66±0.21	0.65±0.23	0.62±0.23
QP Reg + 1.4%	0.66±0.19	0.63±0.21	0.65±0.17
QP Reg + 2.5%	0.67±0.12	0.66±0.11	0.65±0.10
QP Reg + 4.6%	0.71±0.08	0.70±0.07	0.68±0.08
QP Reg + 8.7%	0.75±0.05	0.75±0.06	0.75±0.05
QP Reg + 16.1%	0.83±0.05	0.83±0.05	0.82±0.05
QP Reg + 29.7%	0.92±0.03	0.91±0.03	0.91±0.03
QP Reg + 54.3%	0.98±0.01	0.97±0.01	0.97±0.02
QP Reg + 100.0%	1.00±0.00	1.00±0.00	0.99±0.00

Table 7: Median \pm stdev. on Kendall’s τ between the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

	No noise	Noise within spectral gap	Large noise
True	1.00	1.00	1.00
Spectral	1.00	0.99	0.98
QP	1.00	0.97	0.98
QP Reg	1.00	0.97	0.97
QP Reg add 0.1% cons	0.98	0.95	0.97
QP Reg add 0.7% cons	0.89	0.94	0.88
QP Reg add 1.4% cons	0.85	0.85	0.91
QP Reg add 2.5% cons	0.91	0.86	0.83
QP Reg add 4.6% cons	0.83	0.89	0.85
QP Reg add 8.7% cons	0.91	0.92	0.86
QP Reg add 16.1% cons	0.95	0.93	0.94
QP Reg add 29.7% cons	0.99	0.98	0.98
QP Reg add 54.3% cons	1.00	1.00	1.00
QP Reg add 100.0% cons	1.00	1.00	1.00

Table 8: Best Kendall's τ between the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

	No noise	Noise within spectral gap	Large noise
True	0 \pm 0	142 \pm 99	823 \pm 250
Spectral	0 \pm 0	780 \pm 528	1715 \pm 560
QP	1782 \pm 917	1640 \pm 754	1746 \pm 459
QP Reg	1766 \pm 919	1566 \pm 746	1734 \pm 455
QP Reg + 0.1%	1738 \pm 690	2035 \pm 596	1942 \pm 442
QP Reg + 0.7%	1886 \pm 535	1998 \pm 529	2164 \pm 392
QP Reg + 1.4%	1982 \pm 546	2160 \pm 476	2308 \pm 393
QP Reg + 2.5%	1948 \pm 484	2048 \pm 430	2352 \pm 364
QP Reg + 4.6%	1818 \pm 381	1934 \pm 391	2246 \pm 368
QP Reg + 8.7%	1660 \pm 325	1757 \pm 318	2105 \pm 319
QP Reg + 16.1%	1157 \pm 307	1279 \pm 329	1740 \pm 360
QP Reg + 29.7%	547 \pm 225	780 \pm 264	1278 \pm 305
QP Reg + 54.3%	150 \pm 100	322 \pm 130	932 \pm 275
QP Reg + 100.0%	0 \pm 0	142 \pm 99	798 \pm 251

Table 9: Median \pm stdev. on number of violated R-matrix monotonicity constraints for the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

	No noise	Noise within spectral gap	Large noise
True	0	23	352
Spectral	0	89	414
QP	0	261	488
QP Reg	0	263	547
QP Reg add 0.1% cons	115	439	916
QP Reg add 0.7% cons	655	456	1060
QP Reg add 1.4% cons	822	1064	1139
QP Reg add 2.5% cons	587	1169	1555
QP Reg add 4.6% cons	1002	902	1507
QP Reg add 8.7% cons	750	710	1379
QP Reg add 16.1% cons	336	534	997
QP Reg add 29.7% cons	85	205	615
QP Reg add 54.3% cons	0	54	393
QP Reg add 100.0% cons	0	23	308

Table 10: Best number of violated R-matrix monotonicity constraints for the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

	No noise	Noise within spectral gap	Large noise
True	40145±0	40136±402	40588±4203
Spectral	40145±0	41417±1542	45765±5886
QP	43245±3548	43272±3081	45467±5008
QP Reg	43135±3448	43363±3073	45467±4946
QP Reg + 0.1%	45701±3700	46315±3352	46463±5582
QP Reg + 0.7%	47510±3870	47396±3800	49116±6349
QP Reg + 1.4%	48887±4029	48765±3933	49798±6040
QP Reg + 2.5%	47525±3705	48117±3805	50061±6053
QP Reg + 4.6%	47554±3070	47220±3166	49345±5687
QP Reg + 8.7%	45716±2652	46171±2559	47500±5606
QP Reg + 16.1%	43782±1985	43889±2545	45087±5140
QP Reg + 29.7%	41518±1148	41806±1376	42566±4423
QP Reg + 54.3%	40338±357	40409±500	41004±4230
QP Reg + 100.0%	40145±0	40136±402	40587±4201

Table 11: Median \pm stdev. on objective value in problem (1) for the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

	No noise	Noise within spectral gap	Large noise
True	40145	39032	30820
Spectral	40145	39516	33805
QP	40145	39736	32935
QP Reg	40145	39736	32918
QP Reg add 0.1% cons	40220	40764	33158
QP Reg add 0.7% cons	41562	40642	32337
QP Reg add 1.4% cons	41554	42688	33472
QP Reg add 2.5% cons	41240	42012	33942
QP Reg add 4.6% cons	42397	41892	34176
QP Reg add 8.7% cons	41863	41453	34285
QP Reg add 16.1% cons	40558	40799	32346
QP Reg add 29.7% cons	40173	39523	31998
QP Reg add 54.3% cons	40145	39032	30820
QP Reg add 100.0% cons	40145	39032	30818

Table 12: Best objective value in problem (1) for the true Markov chain ordering, the Fiedler vector, the seriation QP in (6) and the semi-supervised seriation QP in (7) with varying numbers of pairwise orders specified. We observe the (randomly ordered) model covariance matrix (no noise), the sample covariance matrix with enough samples so the error is smaller than half of the spectral gap, then much fewer samples (Large noise).

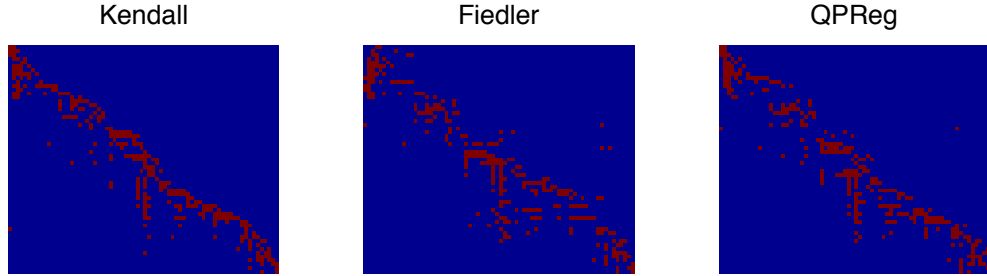


Figure 3: The Hodson's Munsingen dataset: the first figure on the left has the order of the rows given by Kendall, the middle figure is the Fiedler solution, the figure on the right is the best QP solution from 100 experiments with different Y (based on combinatorial objective).

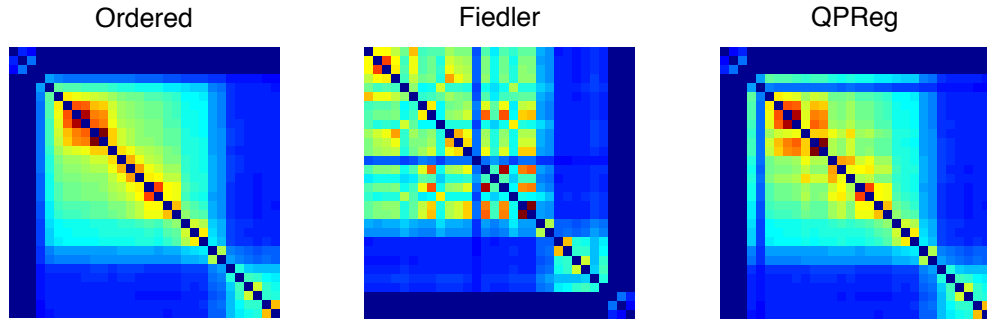


Figure 4: Markov Chain experiments: the first figure on the left has the true order of the Markov chain, the middle figure is the Fiedler solution, the figure on the right is the best QP solution from 100 experiments with different Y (based on combinatorial objective).