

# Supplementary Material for: Learning Halfspaces with the Zero-One Loss: Time-Accuracy Tradeoffs

## A Additional Proofs

### A.1 Proof of Lemma 4

By definition of  $\beta$  we have that  $p(\gamma) = p(1) = 1$  therefore  $p(z) \geq 1$  for all  $z \in [\gamma, 1]$ . The maxima of our polynomial in  $[\gamma, 1]$  is attained at  $z_{\max} = \sqrt{\frac{a_1}{-3a_3}} = \sqrt{\frac{1}{3}[1 + \gamma + \gamma^2]} \in \left(\frac{1}{\sqrt{3}}, \frac{1+\gamma}{\sqrt{3}}\right)$  and its value is

$$\begin{aligned}
 p(z_{\max}) &= \left(\frac{1}{\gamma} + \frac{\gamma}{1+\gamma}\right) z_{\max} - \frac{1}{\gamma(1+\gamma)} z_{\max}^3 \\
 &\leq \left(\frac{1}{\gamma} + \frac{\gamma}{1+\gamma}\right) \frac{1+\gamma}{\sqrt{3}} - \frac{1}{\gamma(1+\gamma)} \frac{1}{3\sqrt{3}} \\
 &= \frac{1}{\gamma(1+\gamma)\sqrt{3}} \left[ (1+\gamma+\gamma^2)(1+\gamma) - \frac{1}{3} \right] \\
 &= \frac{1}{\gamma(1+\gamma)\sqrt{3}} \left[ \frac{2}{3} + 2(\gamma + \gamma^2) + \gamma^3 \right] \\
 &\leq \frac{2}{3\sqrt{3}} \frac{1}{\gamma} + \frac{2}{\sqrt{3}}.
 \end{aligned}$$

Finally, the 1-norm is  $\|\beta\|_1 = \frac{2+\gamma+\gamma^2}{\gamma(1+\gamma)} < \frac{2}{\gamma} + 1$ .

### A.2 Proof of Lemma 7

We will take  $p(z) = \alpha \operatorname{erf}(c\tau z)$ , where  $\operatorname{erf}$  is the error function and  $c = \alpha \operatorname{erf}^{-1}(1/\alpha)$ . By a standard fact,  $\operatorname{erf}$  is equal to its infinite Taylor series expansion at any point, and this series equals

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

Hence,  $p(z)$  is an infinite degree polynomial, and it is only left to verify that the properties stated in the lemma holds for it. Indeed,  $p$  is an odd polynomial and  $|p(z)| \leq \alpha$  for all  $z$ . In addition,

$$p(\gamma) = \alpha \operatorname{erf}\left(\frac{c}{\alpha\gamma}\gamma\right) = \alpha \operatorname{erf}(c/\alpha) = 1.$$

Since  $p$  is monotonically increasing we conclude that  $p(z) \geq 1$  for  $z \geq \gamma$ . Finally, we calculate  $\sum_j \beta_j^2 2^j$ . Since  $c \leq 1$ , we have

$$\begin{aligned}
\sum_{j=3}^{\infty} \beta_j^2 2^j &= \frac{4\alpha^2}{\pi} \sum_{j=1}^{\infty} \left( \frac{(c\tau)^{2j+1}}{j!(2j+1)} \right)^2 2^{2j+1} \\
&\leq \frac{8\alpha^2\tau^2}{\pi} \sum_{j=1}^{\infty} \left( \frac{\tau^{2j}}{j!(2j+1)} \right)^2 2^{2j} \\
&= \frac{8}{\pi\gamma^2} \sum_{j=1}^{\infty} \frac{1}{(2j+1)^2} \left( \frac{(2\tau^2)^j}{j!} \right)^2 \\
&\leq \frac{4}{\pi^2\gamma^2} \sum_{j=1}^{\infty} \frac{1}{j(2j+1)^2} \left( \frac{2e\tau^2}{j} \right)^{2j} \quad \text{Using Stirling's formula} \\
&\leq \frac{4}{\pi^2\gamma^2} \left( \sum_{j=1}^{\infty} \frac{1}{j(2j+1)^2} \right) \cdot \max_j \left( \frac{2e\tau^2}{j} \right)^{2j} \\
&< \frac{0.06}{\gamma^2} e^{4\tau^2},
\end{aligned}$$

where in the last inequality we used the facts that  $\sum_{j \geq 1} \frac{1}{j(2j+1)^2} = \frac{16-\pi^2-4\log 4}{4} < 0.06 \frac{\pi^2}{4}$  and that  $\max_j (2e\tau^2/j)^{2j} \leq e^{4\tau^2}$ . Finally,  $\beta_1^2 2^1 \leq \frac{8}{\pi\gamma^2} < \frac{3}{\gamma^2}$ , hence we conclude our proof.

### A.3 Proof of Lemma 8

Let  $\phi_{\text{sig}}(z) = \frac{1}{1+\exp(-4Lz)}$ . [Shalev-Shwartz et al., 2011, Lemma 2.5] proved the following. For any  $L \geq 3$  and  $\epsilon' \in (0, 1)$ , there exists an odd polynomial,  $g(z) = \sum_j \beta_j z^j$ , such that for all  $z \in [-1, 1]$  we have  $|g(z) - \phi_{\text{sig}}(z)| \leq \epsilon'$  and with

$$\sum_j \beta_j^2 2^j \leq 6L^4 + \exp(9L \log(\frac{2L}{\epsilon'}) + 5).$$

Fix some  $\kappa > 2$  to be specified later. Let  $\bar{\phi}(z) = 2\kappa(\phi_{\text{sig}}(z) - 1/2)$ . It follows that the polynomial  $p(z) = 2\kappa(g(z) - 1/2)$  satisfies

$$|p(z) - \bar{\phi}(z)| = 2\kappa|g(z) - \phi_{\text{sig}}(z)| \leq 2\kappa\epsilon'.$$

Fix some  $\epsilon$  to be also specified later, let  $\epsilon' = \epsilon/(2\kappa)$ , and choose  $L = \frac{1}{4\gamma} \log\left(\frac{\kappa+1}{\kappa-1}\right)$ . By construction,

$$\bar{\phi}(\gamma) = 2\kappa(\phi_{\text{sig}}(\gamma) - 1/2) = 2\kappa\left(\frac{1}{1+\frac{\kappa-1}{\kappa+1}} - 1/2\right) = 2\kappa\left(\frac{\kappa+1}{2\kappa} - \frac{\kappa}{2\kappa}\right) = 1.$$

Therefore, for  $z \geq \gamma$ ,

$$p(z) \geq \bar{\phi}(z) - \epsilon \geq \bar{\phi}(\gamma) - \epsilon = 1 - \epsilon.$$

In addition, for all  $z$ ,  $p(z) \leq \bar{\phi}(z) + \epsilon \leq \kappa + \epsilon$ . Define  $h(z) = p(z)/(1 - \epsilon)$ . So, for  $z \geq \gamma$ ,  $h(z) \geq 1$ , and for any other  $z$ ,  $h(z) \leq \frac{\kappa+\epsilon}{1-\epsilon}$ .

Using the inequality

$$\forall \kappa > 2, \quad \log\left(\frac{\kappa+1}{\kappa-1}\right) = \log\left(1 + \frac{2}{\kappa-1}\right) \leq \frac{2}{\kappa-1} \leq \frac{4}{\kappa}$$

we obtain that  $L \leq \frac{1}{\gamma\kappa}$ .

Now, let's specify  $\kappa, \epsilon$ . First, choose  $\epsilon = 1/\kappa$ . Second, choose  $\kappa$  so that  $\alpha = \frac{\kappa+\epsilon}{1-\epsilon} = \frac{\kappa^2+1}{\kappa-1}$ .

Assume that  $\kappa > 2.5$ , we have that  $1/\kappa \leq 2/\alpha$ . Hence,  $L \leq \frac{2}{\gamma\alpha} = 2\tau$ , which yields the bound,

$$\begin{aligned} B &\leq \left(\frac{2\kappa}{1-\epsilon}\right)^2 (6L^4 + \exp(9L \log(\frac{2L}{\epsilon'}) + 5)) \\ &\leq 4\alpha^2 (96\tau^2 + \exp(18\tau \log(8\tau\alpha^2) + 5)) . \end{aligned}$$

Finally, the assumptions on  $\alpha$  and  $\gamma$  imply that  $\kappa > 2.5$  and that  $L \geq 3$  as required.