Unlocking Fairness: a Trade-off Revisited

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Abstract

The prevailing wisdom is that a model's fairness and its accuracy are in tension with one another. However, there is a pernicious *modeling-evaluating dualism* bedeviling fair machine learning in which phenomena such as label bias are appropriately acknowledged as a source of unfairness when designing fair models, only to be tacitly abandoned when evaluating them. We investigate fairness and accuracy, but this time under a variety of controlled conditions in which we vary the amount and type of bias. We find, under reasonable assumptions, that the tension between fairness and accuracy is illusive, and vanishes as soon as we account for these phenomena during evaluation. Moreover, our results are consistent with an opposing conclusion: fairness and accuracy are sometimes in accord. This raises the question, *might there be a way to harness fairness to improve accuracy after all?* Since many notions of fairness are with respect to the model's predictions and not the ground truth labels, this provides an opportunity to see if we can improve accuracy by harnessing appropriate notions of fairness over large quantities of *unlabeled* data with techniques like posterior regularization and generalized expectation. We find that semi-supervision improves both accuracy and fairness while imparting beneficial properties of the unlabeled data on the classifier.

1 Introduction

Torrents of ink have been spilled characterizing the relationship between a classifier's "fairness" and its accuracy [11, 7, 3, 8, 20, 4, 14, 2, 13, 17], where fairness refers to a concrete mathematical embodiment of some rule provided by an external party such as a government and which must be imposed on a learning algorithm. The consensus, countenanced by both empirical and analytical studies, is that the relationship is a trade-off: satisfying the supplied fairness constraints is achieved only at the expense of accuracy. On the one hand, these findings are intuitive: if we think of fairness as constraints limiting the set of possible classification assignments to those that are collectively fair, then clearly accuracy suffers because in general, optimization over the subset always lower bounds optimization over the original set. As put in another paper "demanding fairness of models *always* come at a cost of reduced accuracy" [2].[1]

On the other hand, the belief in a simple assumption immediately calls these findings into question. In particular, it requires no stretch of credulity to imagine that various personal attributes (e.g., race, gender, religion; sometimes termed "protected attributes") have no bearing on a person's intelligence, capability, potential, qualifications, etc., and consequently no bearing on ground truth classification labels — such as job qualification status — that might be functions of these qualities.[2] It then follows that enforcing fairness across these attributes should on average *increase* accuracy. The reason is clear. If our classifier produces different label distributions depending on the values of these dimensions, then we know, under the foregoing assumption, that at least one of these distributions must be wrong, and thus there is an opportunity to improve accuracy. An opportunity to which we later return.

[1] Our emphasis.

[2] This assumption is consistent with the "we're all equal" worldview [9]

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But first we must understand what accounts for this antinomy. Two possible explanations involve the phenomena of label bias and selection bias. Label bias occurs when the process that produces the labels (e.g., a manual annotation process or a decision making process such as hiring) are influenced by factors that are not particularly germane to the determination of the label value, and thus might differ from the ideal labels, whatever they should have been. Accuracy measured against any such biased labels should be considered carefully with a grain of salt. Selection bias occurs when selecting a subsample of the data in such a way that happens to introduce unexpected correlations, say, between a protected attribute and the target label. Training data, which is usually derived via selection from a larger set of unlabeled data and subsequently frozen in time, is especially prone to this problem.

If pressed to couch the above discussion in a formal framework such as probably approximately correct (PAC) learning, we would say that we have a data distribution $\mathcal{D}$ and labeling function f, either of which could be biased. For example, due to selection bias we might have a flawed data distribution $\mathcal{D}'$ and due to label bias we might have a flawed labeling function f'. This leads to four regimes: the data distribution is biased ($\mathcal{D}'$) or not ($\mathcal{D}$) and the labeling function is biased (f') or not (f). Many theoretical works in fair machine learning consider the regime in which neither is biased, and many empirical works—due in part to the unavailability of an unbiased f—draw conclusions assuming the regime in which neither is biased. But many forms of unfairness arise exactly because one or both of these are biased: hence the dualism in fair machine learning. In this work, we assume that some of the unfairness might arrise because we are actually in one of the other three regimes.

In this paper we account for both label and selection bias in our evaluations and show that when taken into consideration, that certain definitions of fairness and accuracy are not always in tension. Since we do not have access to the unbiased, unobserved ground truth labels in practice, we instead simulate datasets in tightly controlled ways such that, for example, it exposes the actual unbiased labels for evaluation. Encouraged by theoretical results on semi-supervised PAC learning that state that these techniques will be successful exactly when there is compatibility between some semi-supervised signal and the data distribution [1] and the success of GE [16, 10], we also introduce and study a semi-supervised method that exploits fairness constraints expressed over large quantities of unlabeled data to build better classifiers. Indeed, we find that as fairness improves, so does accuracy. Moreover, we find that like other fairness methods, the semi-supervised approach can successfully overcome label bias; but unlike other fairness methods, it can also overcome selection bias on the training set.

2 Related work

Somehow, the idea that fairness and accuracy are not always in tension is both obvious and inconspicuous (but nevertheless of practical significance). The idea appears obvious because we assume the unobserved unbiased ground-truth to be fair, and then limit our hypotheses to the fair region of the space, and then claim that fairness improves accuracy. At this level of generality, it even appears to beg the question, but note that not all fair hypotheses are accurate since in the case we consider a perfectly random classifier is also fair. Moreover, the noise on the observed biased labels with which we train the classifier is diametrically opposed to the unobserved label. Thus even under our assumptions, it is not a foregone conclusion that improving fairness improves accuracy. Rather, our assumption merely leaves open the possibility for this to happen. The finding is inconspicuous in the sense that, as mentioned earlier, there is a preponderance of work investigating this trade-off yet label bias appears to have gone unnoticed: very few papers (e.g., [8, 20]) mention the fact that the labels against which we evaluate are often biased (unfairly against a protected attribute) in the very same way as the unfair classifier trained on them [11, 7, 3, 8, 20, 4, 14, 2, 13, 17]. It may be the case that label-bias is so obvious to most authors that it does not even occur to them to mention it; howbeit, the conspicuous absence of label-bias from papers on fairness perniciously pervades real-world discussions underlying the decisions about how to balance the trade-off between fairness and accuracy. Thus, we believe this finding to be of practical importance and worthy of highlighting.

While uncommon, some papers do indeed mention label-bias, including recent work that considers the largely hypothetical case: if we have access to unbiased labels, then we can propose a better way of evaluating fairness with "disparate mistreatment" [20]. However, their emphasis is on new fairness metrics, not on its tradeoff with accuracy. Other work mentions the problem of label bias in passing, lamenting that it is difficult to account for in practice because we "only have biased data" and thus we "cannot evaluate our classifiers against an unbiased ground truth" and so achieving parity requires

that "one must be willing to reduce accuracy" ([8]). They overcome the lack of unbiased labels via data simulation, a strategy we also employ.

Congruent with our findings, others have noted that the fairness-accuracy tension is not as bad as it seems. Recent work correctly remarks that while there is a tradeoff between fairness and goodness of fit on the training set, that "it does not [necessarily] introduce a tension" since a reduction in model complexity via fairness constraints might act as a regularizer and improve generalization [2]. This is a very interesting remark, but it could have gone even further and addressed generalization with respect to the unbiased labels, which we study in this work.

In recent theoretical work, the authors' propose a "construct space" in which the observed data might differ from some unobserved actual truth about the world [9]. While they investigate many different notions of fairness, they do not address accuracy. The construct space provides a promising theoretical framework for our work, but we save such analysis for another day. Other analytical work studies the trade-off between fairness and accuracy as a function of the amount of statistical dependence between the target class and protected attribute, concluding that only "in the other extreme" of perfect independence that "we can have maximum accuracy and fairness simultaneously" [17]. This "extreme" is none other than the "we're all equal" assumption, which we believe to be perfectly reasonable in many situations. Further, note that this theoretical "maximum" may not be achievable in practice due to imperfect classifiers trained on incomplete, noisy data, or in the context of the phenomena mentioned herein, and hence there is still an opportunity to improve both.

It is worth thinking about the problems of selection and label bias with respect to an existing fairness datasets such as COMPAS, for which the labels are sometimes treated as if they are the unbiased ground truth [20]. Consider that the people in the COMPAS data had been selected from a specific county in Florida with its concomitant pattern of policing, during a specific period of time (2013-2014), meeting a specific set of criteria such as being scored during a specific stage within the judicial system. Each one of these "selections" opens the door for selection bias to introduce unintentional correlations. Indeed, recent work demonstrates that the data is skewed with respect to age, which acts as a confounding variable in existing analysis [18]. Moreover, while not exactly label-bias, the variable indicating recidivism is only partially observed since it considers only a two-year window and assumes that no crime goes uncaught.

Finally, we emphasize that our findings do not imply that the existing theories and conclusions discussed in the literature are incorrect. On the contrary, these works are in fact both sound and relevant. The different conclusions then are explained by the consideration of different types of data bias (or lack thereof) as well as the underlying assumptions, and our assumptions may not always apply [3]. If there differences between groups based on a protected attribute (e.g., due to selection bias), then enforcing fairness could indeed hurt accuracy. We do not address the degree to which one assumption applies to a particular problem or dataset in this paper. Thus, just like in statistical significance testing, it remains up to the discretion of the discerning practitioner to determine if our (or their) set assumptions reasonably apply to the situation in question, and if the assumptions do not, then our (or their) conclusions do not apply, and should be properly rejected as irrelevant to that data.

3 Background

Fairness and bias types We consider two types of biases that lead to unfair machine learning models: label bias and selection bias. Label bias is when the observed binary class labels, say, on the training and testing set, are influenced by protected attributes. For example, the labels in the dataset might be the result of yes/no hiring decisions for each job candidate. It is known that this hiring process is sometimes biased with respect to protected attributes such as race, age or gender. Since decisions might be influenced by protected attributes that on the contrary should have no bearing on the class label, this implies there is a hypothetical set of latent unobserved labels corresponding to decisions that were not influenced by these attributes. We notate these unobserved unbiased labels as z. We notate the observed biased labels as y. Typically, we only have access to the latter for training and testing our models.

Selection bias (skew) occurs when the method employed to select some subset of the overall population biases or skews the subset in unexpected ways. This can occur if selecting based on some attribute that inadvertently correlates with a protected class or the target labels. Training sets are particularly vulnerable to such bias because, for the sake of manual labeling expedience, they are

meager subsamples of the original unlabeled data points. Moreover, this problem is compounded since most available labeled datasets are statically frozen in time and are thus also selectionally biased along the axis of time. For example, in natural language processing (NLP), the particular topical subjects or the entities mentioned in newswire articles change over time: the entities discussed in political discourse today are very different from a decade ago and new topics must emerge to keep pace with the dernier cri [19]. And, as we continue to make progress in reducing discrimination, the discrepancy between the training data of the past and the available data of the present will increasingly differ w.r.t. to selection bias. Indeed, selection bias might manifest itself in a way such that on the relatively small training set, the data examples that were selected for labeling happen to show bias against the protected class. It is with this manifestation of selection bias that we are most concerned, and that we study in the current work.

Illustrative example: learning fair sectors Consider the problem of learning circular sectors of the unit disk with the following attributes: the domain set $\mathcal{X}$ is the unit disk, the label set $\mathcal{Y}$ is $\{0, 1\}$, the data generation model $\mathcal{D}$ is an arbitrary density on $\mathcal{X}$, the labeling function f is an arbitrary partition of $\mathcal{X}$ into two circular sectors, the hypothesis class $\mathcal{H}$ is the set of all partitions of $\mathcal{X}$ into two circular sectors. Samples from $\mathcal{D}$ are points on the unit disk with location $re^{i\phi}$ where $\phi \in [0, 360)$ and $r \in [0, 1]$. We represent a circular sector as a pair of angles (μ, θ) and defined as the circular sector from angle $(\mu - \theta)\%360$ to angle $(\mu + \theta)\%360$ that contains the point $e^{i\mu}$. The labeling function f partitions the disk in two circular sectors $f^{-1}(0)$ and $f^{-1}(1)$ and we will refer to the former as the negative circular sector and the latter as the positive circular sector. Note that for any labeling function f, we have $f \in \mathcal{H}$ and so the realizable assumption holds.

Due to label bias, the labeling function f might be biased (f') as shown in Figure 1. Here, the total positive area according to f is given by the area in green and red, but because of label bias f' only considers points in green as positive. Hence, as demonstrated in Figure 2, an empirical risk minimization (ERM) algorithm will learn a sector (dotted lines) that appears accurate with respect to f', but is much less accurate with respect to f. If we had prior knowledge that the ratio of the positive sector and negative sector should be some constant k, perhaps we could exploit this and improve the ERM solution. We might term such an alternative empirical fairness maximization (EFM) (or fair ERM [5]), and in this paper, we present a semi-supervised EFM algorithm to exploit such fairness knowledge as a constraint on unlabeled data. This example is fully developed in appendix B.

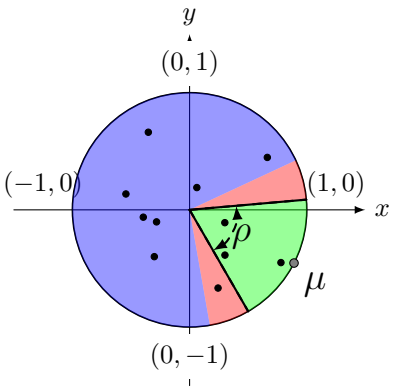


Figure 1

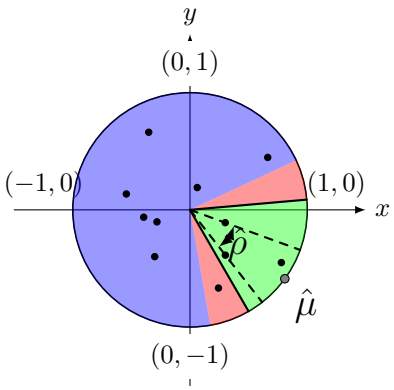


Figure 2

Semi-supervised classification A binary classifier[3] $g_w : \mathbb{R}^k \to \{0, 1\}$ parameterized by a set of weights $w \in \mathbb{R}^k$ is a function from a k dimensional real valued feature space, which is often in practice binary, to a binary class label. A probabilistic model $p_w(\cdot|x)$ parameterized by (the very same) w underlies the classifier in the sense that we perform classification by selecting the class label (0 or 1) that maximizes the conditional probability of the label y given the data point x

$$g_w(x) = \underset{y \in \{0,1\}}{\text{argmax}}\, p_w(y|x) \tag{1}$$

[3]For ease of explication, we consider the task of binary classification, though our method can easily be generalized to multiclass classification, multilabel classification, or more complex structured prediction settings.

We can then train the classifier in the usual supervised manner by training the underlying model to assign high probability to each observed label y_i in the training data $\mathcal{D}_{\text{tr}} = \{\langle x_i, y_i\rangle \mid i = 1 \dots n\}$ given the corresponding example x_i, by minimizing the negative log likelihood:

$$\hat{w} = \underset{w \in \mathbb{R}^k}{\text{argmin}} \sum_{\langle x_i, y_i\rangle \in \mathcal{D}_{\text{tr}}} -\log p_w(y_i|x_i) \tag{2}$$

We can extend the above objective function to include unlabeled data $\mathcal{D}_{\text{un}} = \{x_i\}_{i=1}^n$ to make the classifier semi-supervised. In particular, we add a new term to the loss, $\mathcal{C}(\mathcal{D}_{\text{un}}, w)$, with a weight η to control the influence of the unlabeled data over the learned weights:

$$\hat{w} = \underset{w \in \mathbb{R}^k}{\text{argmin}} \left(\sum_{\langle x_i, y_i\rangle \in \mathcal{D}_{\text{tr}}} -\log p_w(y_i|x_i) \right) + \eta \mathcal{C}(\mathcal{D}_{\text{un}}, w) \tag{3}$$

The key question is how to define the loss term $\mathcal{C}$ over the unlabeled data in such a way that improves over our classifier.

4 Approach

Apropos the foregoing discussion, we propose to employ fairness in the part of the loss function that exploits the unlabeled data. There are of course many definitions of fairness proposed in the literature that we could adapt for this purpose, but for now we focus on a particular type of group fairness constraint derived from the *statistical parity* of selection rates. Although this definition has (rightfully) been criticized, it has also (rightfully) been advocated in the literature and it underlies legal definitions such as the 4/5ths rule in U.S. law [6, 8, 21]. For the purpose of this paper, we do not wish to enter the fray on this particular matter.

More formally, let $S = \{x_i\}_{i=1}^n$ be a set of n unlabeled examples, then the selection rate of the classifier g_w is $\bar{g}_w(S) = \frac{1}{n}\sum_{x_i \in S} g_w(x_i)$. If we partition our data ($\mathcal{D}_{\text{un}}$) into the protected ($\mathcal{D}_{\text{un}}^P$) and unprotected ($\mathcal{D}_{\text{un}}^U$) partitions such that $\mathcal{D}_{\text{un}} = \mathcal{D}_{\text{un}}^P \cup \mathcal{D}_{\text{un}}^U$, then we want the selection rate ratio

$$\frac{\bar{g}_w(\mathcal{D}_{\text{un}}^P)}{\bar{g}_w(\mathcal{D}_{\text{un}}^U)} \tag{4}$$

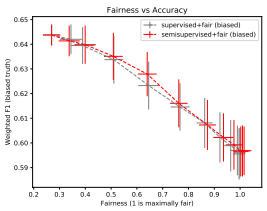
to be as close to one as possible. However, to make the problem more amenable to optimization via stochastic gradient descent, we relax this definition of fairness to make it differentiable with respect to w. In particular, analogous to $\bar{g_w}(S)$, define $\bar{p}_w(S) = \frac{1}{n}\sum_{x_i \in S} p_w(y = 1|x_i)$ to be the average probability of the set when assigning each example x_i to the positive class $y_i = 1$. Then, the group fairness loss over the unlabeled data — which when plugged into Equation 3 yields an instantiation of the proposed semisupervised training technique discussed herein — is

$$\mathcal{C}(\mathcal{D}_{\text{un}}, w) = \left(\bar{p}_w(\mathcal{D}_{\text{un}}^P) - \bar{p}_w(\mathcal{D}_{\text{un}}^U)\right)^2 \tag{5}$$

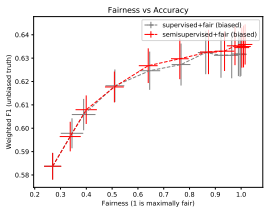
Parity is achieved at zero, which intuitively encodes that overall, the probability of assigning one group to the positive class should on average be the same as assigning the other group to the positive class. This loss has the important property that it is differentiable with respect to w so we can optimize it with stochastic gradient descent, along with the supervised term of the objective, making it easy to implement in existing toolkits such as Scikit-Learn, PyTorch or TensorFlow.

5 Experiments

In this section we investigate the relationship between fairness and accuracy under conditions in which we can account for (and vary) the amount of label bias, selection bias, and the extent to which the classifiers enforce fairness. Typically, accuracy is measured against the ground truth labels on the test set, which inconspicuously possesses the very same label bias as the training set. In this typical evaluation setting, if we train a set of classifiers that differ only in the extent to which their training objective functions enforce fairness, and then record their respective fairness and accuracy scores on a test set with such label bias, we see that increased fairness is achieved at the expense



(a) COMPAS (biased ground truth)



(b) COMPAS (unbiased ground truth)

Figure 3: Accuracy vs. fairness on simulated (β=0.25) COMPAS (assumption hold).

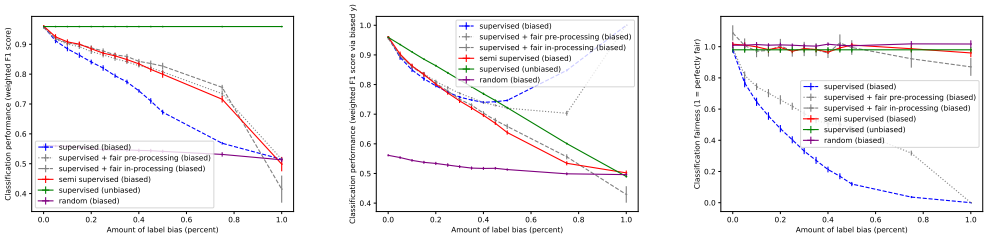
of accuracy (Figure 3a). However, because the labels are biased, we must immediately assume that the corresponding accuracy measurements are also biased. Therefore, we are crucially interested in evaluating accuracy on the *unbiased ground truth labels*, which are devoid of any such label bias. Since we do not have access to the unbiased ground truth labels of real-world datasets, we must instead rely upon data simulation. We discuss the details later, but for now, assume we could evaluate on such data. In Figure 3a, we evaluate the same set of classifiers as before, but this time measure accuracy with respect to the unbiased ground truth labels. We see the exact opposite pattern: classifiers that are more fair are also more accurate. With the gist of our results and experimental strategy in hand, we are now ready to describe the assumptions, data simulator, and systems to undertake a more comprehensive empirical investigation.

Assumptions We make a set of assumptions that we encode directly into the probabilistic data generator, explained in more detail below. For example, we encode the "we're all equal assumption" by making the unbiased labels statistically independent of the protected class [9]. If these assumptions do not hold in a particular situation, then our conclusions may not apply. We describe the assumptions in more detail below and in the appendix.

Data Our experiments require datasets with points of the form $\mathcal{D} = \{x, \rho, z, y\}$ in which x is the vector of unprotected attributes, ρ is the binary protected attribute, z is the (typically unobserved) label that has no label bias and y is the (typically observed) label that may have label bias. Since z is unobserved — and even if it were available, we would still want to vary the severity of label bias for experimental evaluation — we must rely upon data simulation [8]. We therefore assume that the biased labels are generated from the unbiased labels via a probabilistic model g and assume that $y \sim g(y|z, \rho, x, \beta)$ where β is a parameter of the model that controls the probability of label bias occurring. Now we have two options for generating datasets of our desired form, we can either (a) simulate the dataset entirely from scratch from a probabilistic model of the joint distribution $P(x, \rho, z, y) = g(y|z, \rho, x, \beta)P(z, \rho, x)P(\beta)$, or we can (b) begin with an existing dataset, declare by fiat that the labels have no label bias (and are thus observed after all) and then augment the data with a set of biased labels sampled from $g(y|z, \rho, x, \beta)$.

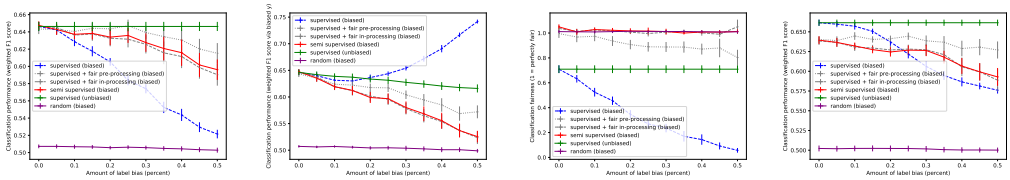
For data of type (a) we generate the features and labels (both biased and unbiased) entirely from scratch with the Bayesian network in Figure 7 (Appendix A.2). For this data, we explicitly enforce the following statistical assumptions: $z, x \perp\!\!\!\perp \rho$, $y \not\perp\!\!\!\perp \rho$, $z \not\perp\!\!\!\perp x$, $y \not\perp\!\!\!\perp z$. A parameter β controls the amount of label bias; σ controls the amount of selection bias, which can break some assumptions. For data of type (b) we begin with the COMPAS data, treat the two-year recidivism labels as the unbiased ground-truth z and then apply our model of label bias to produce the biased labels $y \sim g(z|y, \rho, x, \beta)$ [15]. Since the "we're all equal" assumption does not hold for COMPAS data we also create a second type of test data in which we enforce demographic parity via subsampling so that our assumption holds (see Appendix A.3).

Systems, baselines and evaluations We study the behavior of the following classification systems. A traditional supervised classifier trained on biased label data, a supervised classifier trained on unbiased label data (this in some sense is an ideal model, but not possible in practice because we do not have access to the unbiased labels in practice), a random baseline in which labels are sampled according to the biased label distribution in the training data, and three fair classifiers. The first fair classification method is an in-processing classifier that employs our fairness constraint, but as a regularizer on the training data instead of the unlabeled data. The resulting classifier is similar



(a) F1 (unbiased truth) (b) F1 (biased truth) (c) Fairness

Figure 4: Classifier accuracy (F1) and fairness as a function of the amount of label bias.



(a) F1 (unbiased truth) (b) F1 (biased truth) (c) Fairness (d) F1 (unbiased truth (v))

Figure 5: Varying label bias on COMPAS (assumption holds, except in 5d).

to the prejudice remover, but with a slightly different loss [12]. The second fair classifier is a supervised logistic regression trained using the "reweighing" pre-processing method [11]. The final fair classifier, which we introduce in this paper, is a semi-supervised classifier that utilizes the fairness loss (Equation 5) on the unlabeled data.

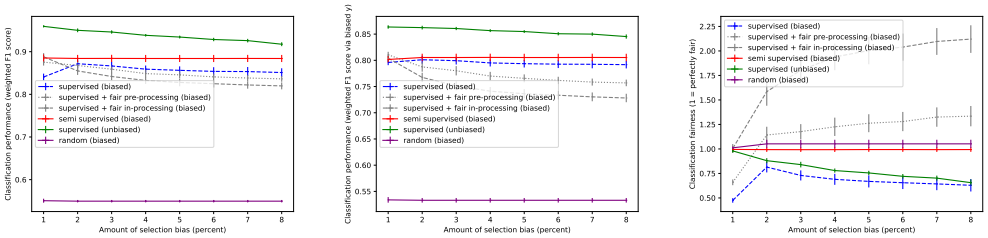
We assess fairness with a group metric that computes the ratio of the selection rates of the protected and unprotected class, as we defined in Equation 4. A score of one is considered perfectly fair. To assess 'accuracy' we compute the weighted macro F1, which is the macro average weighted by the relative portion of examples belonging to the positive and negative classes. We evaluate F1 with respect to both the biased labels and the unbiased labels. We always report the mean and standard error of these various metrics computed over ten experiments with ten randomly generated datasets (or in the case of COMPAS, ten random splits).

5.1 Experiment 1: Label Bias

In this experiment we investigate the relationship between fairness and accuracy for each classification method as we vary the amount of label bias. All classifiers except the unbiased baseline are trained on biased labels. If we evaluate the classifiers on the biased labels as in Figure 4b (data simulated from scratch) or Figure 5b (COMPAS data) we see that the classifiers that achieve high fairness (close to one, as seen in Figure 4c&5c) sometimes degrade the (biased) F1 accuracy as commonly seen in the literature. On the other hand, if we evaluate the classifiers on the unbiased labels as in Figure 4a&5a, we see that fairness and accuracy are in accord: the classifiers that achieve high fairness achieve better accuracy than the fairness-agnostic supervised baseline. The gap between the fair and unfair classifiers increases as label bias increases. We also evaluate the classifiers on COMPAS data that violates the "we're all equal" assumption. In this case, the fairness classifiers are enforcing something untrue about the data, and thus fairness initially degrades accuracy (Figure 5d). However, as the amount of label bias increases, eventually there comes a point at which fairness once again improves accuracy (possibly because the amount of label bias exceeds the amount of other forms of bias).

5.2 Experiment 2: Selection Bias

We repeat the experiment from the last section, but this time fixing label bias ($\beta = 0.2$) and subjecting the training data to various amounts of selection bias by lowering the probability that a data example with a positive label is assigned to the protected class. This introduces correlations in the training set between the protected class and the input features as well as correlations with both the unbiased and



(a) F1 (unbiased truth) (b) F1 (biased truth) (c) Fairness

Figure 6: Classifier accuracy (F1) and fairness as a function of the amount of selection bias.

biased labels. These correlations do not exist in the test set and unlabeled set which we assume do not suffer from selection bias. We vary selection bias along the abscissa while keeping the label bias at a constant 20%, and report the same metrics as before. Results are in Figure 6. The main findings are that (a) the results are consistent with the theory that fairness and accuracy are in accord and (b) that the semi-supervised method succesfully harnesses unlabeled data to correct for the selection and label bias in the training data (while the inprocessing fairness method succumbs to the difference in data distribution between training and testing). Let us now look at these findings in more detail and in the context of the other baselines.

Interestingly, the fairness-agnostic classifiers and two of the fairness-aware classifiers (in- and pre-processing) all succumb to selection bias, but in opposite ways (Figure 6c). The fairness-agnostic classifier learns the correlation between the protected attribute and the label and is unfair to the protected class. In contrast, the two supervised fair classifiers, for which fairness is enforced with statistics of the *training set* both learn to overcompensate and are unfair to the unprotected class (its fairness curve is above 1). In both cases, as selection bias increases, so does unfairness and this results in a concomitant loss in accuracy (when evaluated not only against the unbiased labels (Figure 6a), *but also against the biased labels* (Figure 6b)), indicating that fairness and accuracy are in accord. Finally, let us direct our attention to the performance of the proposed semi-supervised method by examining the same figures (Figure 6c). Now we see that regardless of the amount of selection bias, the semi-supervised method successfully harnesses the unbiased unlabeled data to rectify it, as seen by the flat fairness curve achieving a nearly perfect 1 (Figure 6c). Moreover, this improvement in fairness over the supervised baseline (biased trained) is associated with a corresponding increase in accuracy relative to that same baseline (Figures 6a & 6b), regardless of whether it is evaluated with respect to biased (20% label-bias) or unbiased labels (0% label-bias). Note that the "we're all equal" assumption is violated as soon as we evaluate against the biased labels. Moreover, the label-bias induces a correlation between the protected class and the target label, which is a common assumption for analysis showing that fairness and accuracy are in tension [17]. Yet, the beneficial relationship between accuracy and fairness is unsullied by the incorrect assumption in this particular case.

6 Conclusion

We studied the relationship between fairness and accuracy while controlling for label and selection bias and found that under certain conditions the relationship is not a trade-off but rather one that is mutually beneficial: fairness and accuracy improve together. We focused on demographic parity in this paper, but the ideas emphasized in this work, especially label bias, have potentially serious implications for other notions of fairness that go beyond even their relationship with accuracy. In particular, recent ways of assessing fairness such as disparate mistreatment, equal odds and equal oppurtunity involve error rates as measured against labeled data. Label bias raises questions about the reliability of such measures and investigating such questions — about how label bias affects fairness and whether this causes fairness methods to undercompensate or overcompensate — is an important direction of future work. Other future directions would be to develop more complex models of label and selection bias for particular domains so we can better understand the relationship between fairness and accuracy in these domains.

7 Acknowledgements

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A Data Generation

A.1 Assumptions

We make a set of assumptions that we encode directly into the probabilistic data generator, explained in more detail below. If these assumptions do not hold in a particular situation, then our conclusions may not apply. For example, in one experiment we assume no selection bias; in another, we assume the training data experiences more selection bias than the test data. If the opposite were true then, perhaps, a post-processing method of enforcing fairness might be more appropriate. We also make a "we're all equal" assumption [9], which we encode by ensuring that the unbiased labels are statistically independent of the protected dimension. Again, if this assumption is violated to a sufficiently large degree, then our conclusions do not apply. This assumption is important, but not necessary for the finding that fairness and accuracy are not always in tension. Finally, our framework optimistically presupposes that it is possible to model the way in which these biases actually infiltrate real-world datasets. For the purpose of this initial study, we employ the simplest possible models of biases — perhaps at the risk of oversimplifying — that still strongly capture their baleful effects, which researchers in fair ML toil to address.

A.2 Simulated Data (non-COMPAS)

Here we provide details of our data simulation process. In particular, we enforce that the unobserved unbiased labels z_i do not statistically depend on the example's status as protected (or not) ρ_i and only on its other input features x_i. Second, the protected status ρ does not depend on any features x_i. Third, the observed biased labels y_i are biased to depend on the protected status ρ by an amount controlled by the experimental parameter β, which we vary in our experiments.

In summary, we enforce the following statistical properties:

$$z, x \perp\!\!\!\perp \rho \qquad\qquad y \not\perp\!\!\!\perp \rho$$
$$z \not\perp\!\!\!\perp x \qquad\qquad y \not\perp\!\!\!\perp z$$

Where $\perp\!\!\!\perp$ (respectively, $\not\perp\!\!\!\perp$) are the familiar symbols for expressing statistical independence (respectively, dependence) of random variables. Note when we later introduce selection bias, it will break some of these independence assumptions (between ρ and z, x, but in a controlled manner, so we then show to what extent we correct this via unlabeled data, as we vary the amount of selection bias.

To simulate the dataset, we sample the input data points x iid from a k dimensional binary feature space. We sample such that some dimensions contain common features while others contain rare features, in effort to reflect that real-world datasets. In particular, we sample each dimension i according to a Bernoulli proportional to $\frac{1}{i}$ making some dimensions common and others rare.

The parameters of our data generator are β the amount of label bias, τ, which controls the discrepancy between the rarity of features, and α, which controls the ratio between members of the protected and unprotected class. We also introduce a parameter σ that controls the amount of selection bias. First, we sample the observed samples x_i and its status as protected ($\rho = 1$) or not ($\rho = 0$), independently to ensure that protected status and input features are not statistically dependent. Next, we sample the unobserved unbiased labels z from x_i while crucially ignoring the protected status ρ_i to ensure that the label is indeed unbiased. Finally, we sample the observed biased labels y in a way to make them dependent on the class labels ρ_i, a dependency strength controlled by β. More precisely:

$$w_{\text{gen}} \sim N(\mathbf{0}, \Sigma) \tag{6}$$
$$\rho_i \sim \text{Bernoulli}(\alpha) \tag{7}$$
$$x_i^j \sim \text{Bernoulli}\left(\frac{1}{j+1}\right)^{\tau} \text{ for } j = 0, \ldots, k-2 \tag{8}$$
$$z_i = \max(0, \text{sign}\left(w_{\text{gen}}^T x_i\right)) \tag{9}$$
$$y_i \sim g(y|z_i, \rho_i, x_i, \beta) \tag{10}$$

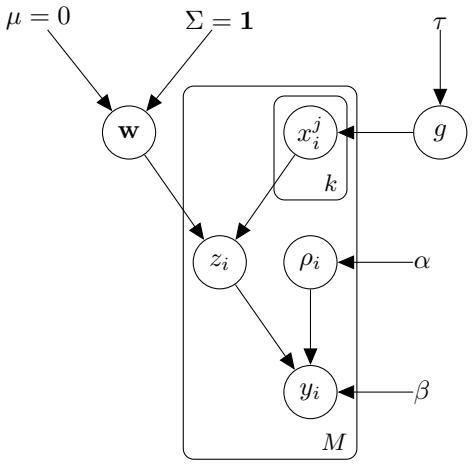

Figure 7: Data generator as a Bayesian network.

where g is the label bias model, parameterized by $\beta \in [0, 1]$, the amount of label bias to introduce, and is a function of the protected dimension and the unobserved unbiased labels z_i, and defined as

$$g(y_i|z_i, \rho_i, x_i, \beta) = g(y_i|z_i, \rho_i, \beta) = \begin{cases} \beta & \text{if} \quad y_i \neq z_i \wedge z = \rho_i \\ 1 - \beta & \text{o.w.} \end{cases} \tag{11}$$

This model assumes that the desirable label is 1 (say calling a candidate for an interview, or offering a loan) and that the bias will be against the protected class and in favor of the unprotected class. Hence with probability β, a protected class individual that has an unbiased label of 1 will have it flipped to 0; similarly, an unprotected class individual that has an unbiased label of 0 will have it flipped favorably to one with probability β. Note the model is simplistic in that it does not make use of the unprotected features and that it assumes symmetry in the bias, as just described. Note that other models could be used for specific datasets or problem domains for which a domain expert has a theory or insight about what the nature of the label bias might be.

The function returns the unbiased labels with probability $1 - \beta$, but otherwise works *against* examples of the protected class by assigning their labels to 0, and *for* all other examples by assigning their labels to 1. See Figure 7 for a Bayesian network representation of the generator.

We can control the amount of selection bias with a parameter σ. As selection bias increases, the data is increasingly unlikely to contain members of the protected attribute ($\rho = 1$) that have a favorable class label ($z = 1$). In particular, if $r \in [0, 1]$ is the portion of the protected attribute assigned to the favorable class in the original data, then the selection bias process "selects" the data in such a way to reduce this portion to $\frac{r}{\sigma}$ for $\sigma \geq 1$. Therefore if $\sigma = 1$ then no bias occurs, and if $\sigma > 1$ then an amount of bias against the protected class occurs proportional to σ. Note that this selection procedure introduces statistical dependencies between the input x and the unbiased label z as well as between the protected class ρ and the unbiased label.

For the non-COMPAS experiments, we generate 20-dimensional binary input features x_i and, 200 training examples, 1000 testing examples and 10,000 unlabeled examples. Note that 200 training examples is reasonable since it means that $n \gg k$ as is usually required for training machine learning models. Yet, at the same time, it is small enough to allow for the weights of some of the rarer features to be under-fit as is typical in most applications of machine learning. Also, unless otherwise stated, the expected protected to unprotected class ratio is even at 50/50, though we have repeated the experiments but with a skewed expected ratio of 20/80 and found it did not affect the conclusions. We train each classifier with 10 epochs of stochastic gradient descent, which we found to be sufficient for this dataset.

A.3 COMPAS Data with Simulated Bias

The COMPAS data is a dataset of criminal recidivism. Here, the task is to predict recidivism (after two-years) from a set of demographic features including age (under 25, over 45 and between 25 and

45), sex, race, prior count (0-37), charge degree (misconduct or felony). We employ race (African-American or not) as the binary protected attribute. A key challenge with real-world data such as COMPAS is that it exhibits both selection and label bias, thus making it difficult to perform our evaluations, which crucially rely on the existence of an *unbiased test set*. To this end, we resort to a combination of simulation and subsampling to unbias the test set.

First, we assume that there is no label bias in the two-year recidvism labels, but then create a biased version of the labels in a simular fashion as before (in this case, by randomly flipping the recidivism label for African Americans from 0 to 1, and by randomly flipping the recidivism label for all others from 1 to 0). In this way, we can create a discrepancy in label bias between the training and testing data. Second, we can force the test set to adhere to the "we're all equal" assumption by subsampling the data such that the recidivism rates are the same for the protected and unprotected class. We thus have two versions of the test data, one in which we enforce this assumption and one in which we do not. In either case, we can vary the amount of label bias on the training set in the same way.

For our COMPAS experiments, we perform ten random splits of the data (7215 total examples) in which we partition the data into 40% train 40% unlabeled and 20% test. Each algorithm is fit with stochastic gradient descent, trained for two epochs on the training data. We found that two epochs was sufficient for training, likely because the training set is large (almost 3000 examples) relative to its dimensionality (about 50 features).

B Learning Circular Sectors: Complete Example

We present the completely developed example mentioned in the paper on learning circular sector with a potentially biased labeling function. This illustrates precisely the intuitive idea that when the labeling function is biased, then ERM's guarantees with respect to the true labeling function are void, while trying to maximize fairness has very strong guarantees.

B.1 Learning Circular Sectors

We consider the problem of learning circular sectors of the unit disk with the following attributes. The domain set $\mathcal{X}$ is the unit disk, the label set $\mathcal{Y}$ is $\{0, 1\}$, the data generation model $\mathcal{D}$ is an arbitrary density on $\mathcal{X}$, the labeling function f is an arbitrary partition of $\mathcal{X}$ into two circular sectors, the hypothesis class $\mathcal{H}$ is the set of all partitions of $\mathcal{X}$ into two circular sectors.

Samples from $\mathcal{D}$ are points on the unit disk with location $re^{i\phi}$ where $\phi \in [0, 360)$ and $r \in [0, 1]$. We represent a circular sector as a pair of angles (μ, θ) and defined as the circular sector from angle $(\mu - \theta)\%360$ to angle $(\mu + \theta)\%360$ that contains the point $e^{i\mu}$. The labeling function f partitions the disk in two circular sectors $f^{-1}(0)$ and $f^{-1}(1)$ and we will refer to the former as the negative circular sector and the later as the positive circular sector. Note that for any labeling function f, we have $f \in \mathcal{H}$ and so the realizable assumption holds.

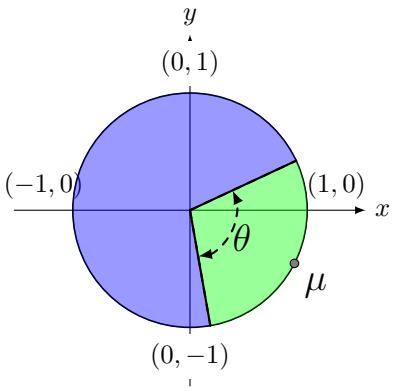

Figure 8: Domain set $\mathcal{X}$ with the positive region of location μ and spread θ in green and the negative region in blue.

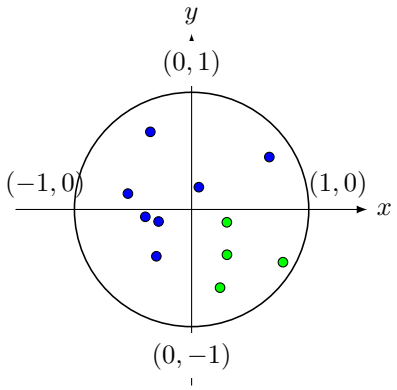

Figure 9: Samples from $\mathcal{D}$ labeled by f.

input : A set $\mathcal{S}$ of N labeled examples $(r_j\ e^{i\phi_j}, l_j)$ for $j \in [N]$
precondition : $\mathcal{S}$ contains at least 3 positive examples with distinct angles
output : A hypothesis $h \in \mathcal{H}$

$a = max\{\phi_j \mid (r_j\ e^{i\phi_j}, l_j) \in \mathcal{S} \wedge l_j = 1\}$;
$b = min\{\phi_j \mid (r_j\ e^{i\phi_j}, l_j) \in \mathcal{S} \wedge l_j = 1\}$;
$c = choose\{\phi_j \mid (r_j\ e^{i\phi_j}, l_j) \in \mathcal{S} \wedge \phi_j \neq a \wedge \phi_j \neq b\}$;
if $b < c < a$ **then**
  $\theta = a - b$;
  $\mu = b + \theta/2$;
else
  $\theta = 360 - a + b$;
  $\mu = a + \theta/2$;
end
return $(e^{i\mu}, \theta)$

Algorithm 1: A_{ERM}, Smallest positive circular sector

The smallest positive circular sector algorithm implements empirical risk minimization and is probably approximately correct.

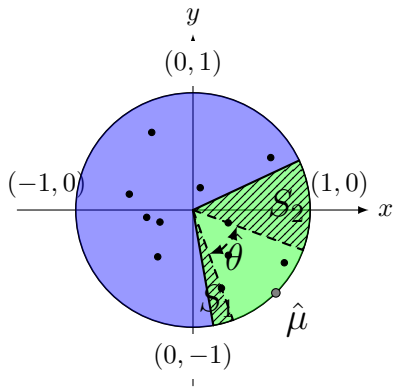

Figure 10: Error of A_{ERM}

Theorem 1 (The A_{ERM} algorithm is probably approximately correct)**.** *For any accuracy $\epsilon > 0$ and confidence $0 < \delta < 1$, there exists a finite number m such that if we take m independent samples*

$e_1, ..., e_m$ *from* $\mathcal{D}$ *and let* $h = A_{ERM}(\{(e_i, f(e_i))\})$ *we have*

$$\Pr\left[\mathcal{D}(\{x \mid h(x) \neq f(x)\}) > \epsilon\right] \leq 1 - \delta$$

Proof. In the following proof, we assume that we have at least 3 examples with distinct angles, which is a fairly mild assumption since the probability of two samples having the same angle is zero.

First note that since all the positive examples belong to the positive circular sector, the smallest positive circular sector returned by the algorithm, h, is a subset of the positive circular sector $f^{-1}(1)$. Indeed, $f^{-1}(1)$ can be partitioned into three circular sectors S_1, S_2, and h such that S_1 and S_2 correspond to the only two area where $h(x) \neq f(x)$ for any x. Therefore, by the additivity of measures

$$\mathcal{D}(\{x \mid h(x) \neq f(x)\}) = \mathcal{D}(S_1) + \mathcal{D}(S_2) \tag{12}$$

For $\mathcal{D}(S_1) + \mathcal{D}(S_2)$ to be greater than ϵ, we must have $\mathcal{D}(S_1) > \epsilon/2$ or $\mathcal{D}(S_2) > \epsilon/2$. Therefore, by the union bound

$$\Pr\left[\mathcal{D}(\{x \mid h(x) \neq f(x)\}) > \epsilon\right] \leq \Pr\left[\mathcal{D}(S_1) > \epsilon/2\right] + \Pr\left[\mathcal{D}(S_2) > \epsilon/2\right] \tag{13}$$

Let us focus on the first term involving S_1. Assume that we have m examples. We know that none of our m examples belong to S_1, or the algorithm would have returned a different, larger, hypothesis. The probability that none of our m samples fell in S_1 is $(1 - \mathcal{D}(S_1))^m$, so the probability that $\mathcal{D}(S_1)$ be greater than $\epsilon/2$ is at least $(1 - \epsilon/2)^m$. The same argument holds for S_2 so

$$\Pr\left[\mathcal{D}(\{x \mid h(x) \neq f(x)\}) > \epsilon\right] \leq 2(1 - \epsilon/2)^m \tag{14}$$

As m increases, $2(1 - \epsilon/2)^m$ decreases, so for any $\delta < 1$, we can choose m such that $2(1 - \epsilon/2)^m$ is smaller than $1 - \delta$. □

This theorem holds for any distribution $\mathcal{D}$ and it is worth noting that this includes distributions which satisfies $\mathcal{D}(S_1) = 0$. In such a case, the selected hypothesis h cannot get close to f regardless of how many samples we draw, but the algorithm is still correct since $\mathcal{D}(\{x \mid h(x) \neq f(x) \wedge x \in S_1\}) = 0$. This remark will be important later on in our presentation.

B.2 Fairness

Assume that we have some predicate $P(.,.)$ which is true on $\mathcal{D}$, and f.

As an example, for some constant k, such a predicate could be defined as

$$\frac{\lambda(f^{-1}(0))}{\lambda(f^{-1}(1))} = k \tag{15}$$

where λ is the Lebesgues measure. This predicate simply states that the ratio of the area of the two circular sectors is a constant.

Pure learning bias. Assuming that we have a training set $\mathcal{S}$ and $h = A_{ERM}(\mathcal{D}, f)$, the probability that $P(\mathcal{S}, h)$ holds is 0. However, because A_{ERM} is probably approximately correct, we know that we can use a training set large enough that $\frac{\lambda(h^{-1}(0))}{\lambda(h^{-1}(1))}$ tends to k.

Data Generation bias. If the samples are drawn from a distribution $\mathcal{D}'$ that is different than $\mathcal{D}$.

Labeling bias. If the samples are labeled by a function f' which is different than f.

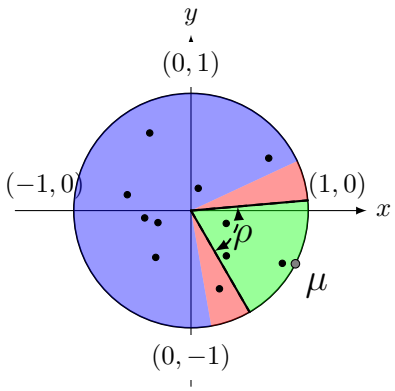

Figure 11: Labeling with f'. The positive region of f' is green, the negative region of f' is blue or red, with the red region indicating where f and f' are different.

There are other predicates we could be interested in

$$\frac{\mathcal{D}(f^{-1}(0))}{\mathcal{D}(f^{-1}(1))} = k \tag{16}$$

C Empirical Risk Minimization

What happens if we use algorithm A_{ERM} on a training set generated with $(\mathcal{D}, f')$? Then, some of the samples which would have been labeled as positive by f are labeled as negative by f'. In consequence, with respect to f', A_{ERM} returns a smaller circular sector than with respect to f.

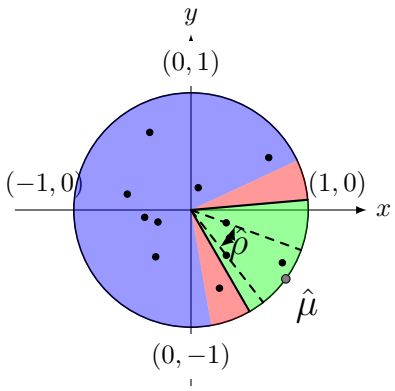

Figure 12: A_{ERM} returns a circular sector of location $\hat{\mu}$ and spread $\hat{\rho}$

This has two implications when we assess the error of A_{ERM}.

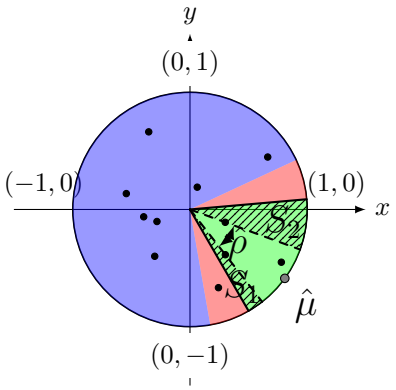

Figure 13: Error of A_{ERM} w.r.t. f'

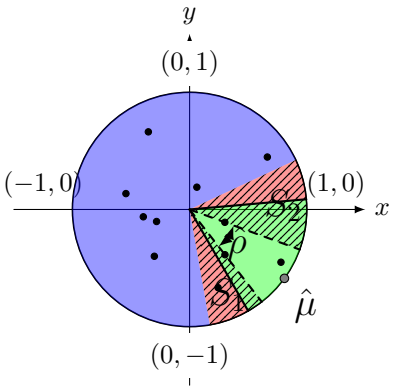

Figure 14: Error of A_{ERM} w.r.t. f

1. With respect to f', A_{ERM} is probably approximately correct
2. With respect to f, A_{ERM} has an error lower bound of $\mathcal{D}(S_1) + \mathcal{D}(S_2)$

D Empirical Risk Minimization with Empirical Fairness Maximization

We now design an algorithm A_{EFM} which is such that for any training set $\mathcal{S}$, we return an hypothesis h for which $P(\mathcal{D}, h)$ holds. Since $P(\mathcal{D}, h)$ must hold, we have

$$\lambda(h^{-1}(1)) = \frac{\lambda(h^{-1}(0))}{k} \tag{17}$$

and we also know

$$\lambda(h^{-1}(0)) + \lambda(h^{-1}(1)) = 2\pi \tag{18}$$

so we conclude

$$\lambda(h^{-1}(1)) = \frac{2\pi}{k+1} \tag{19}$$

and therefore the spread of h must be $360/(k+1)$.

input : A set $\mathcal{S}$ of N labeled examples $(r_j\ e^{i\phi_j}, l_j)$ for $j \in [N]$
precondition : $\mathcal{S}$ contains at least 3 positive examples with distinct angles
output : A hypothesis $h \in \mathcal{H}$
$(e^{i\mu}, .) = A_{ERM}(\mathcal{S})$;
return $(e^{i\mu}, 360/(k+1))$

Algorithm 2: A_{EFM}: Smallest positive circular sector with fairness maximization

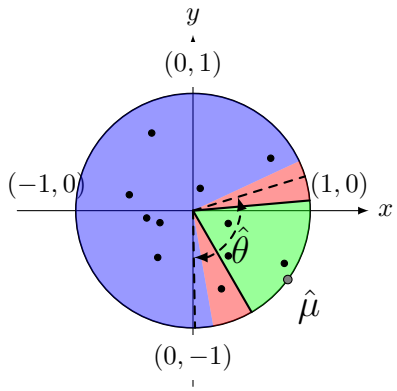

Figure 15: A_{EFM} returns a circular sector of location $\hat{\mu}$ and spread $\hat{\theta}$

First note that A_{EFM} is not probably approximately correct with respect to f'. Indeed, since $\lambda(f'^{-1}(1))$ is smaller than $\lambda(f^{-1}(1))$, $\lambda(f'^{-1}(1)) < 360/(k+1)$.

We will now prove that A_{EFM} is probably approximately correct wirth respect to f. Intuitively this is because h has the right spread by construction and with enough data, we should be able to have a good estimate of the location of f with high probability.

Theorem 2 (Algorithm A_{EFM} is probably approximately correct for positive densities)**.** *For any accuracy $\epsilon > 0$ and confidence $0 < \delta < 1$, if $\mathcal{D}$ is psitive then there exists a finite number m such that if we take m independent samples $e_1, ..., e_m$ from $\mathcal{D}$ and let $h = A_{ERM}(\{(e_i, f(e_i))\})$ we have*

$$\Pr\left[\mathcal{D}(\{x \mid h(x) \neq f(x)\}) > \epsilon\right] \leq 1 - \delta$$

Proof. Since f' is contained in f, the location of f' is in the circular sector of f. Therefore, h and f overlap and their union can be partioned in three different sets, $S_1 = h - f$, $f \cap h$, and $S_2 = f - h$. By the additivity of measures we conclude

$$\mathcal{D}(\{x \mid h(x) \neq f(x)\}) = \mathcal{D}(S_1) + \mathcal{D}(S_2) \tag{20}$$

For $\mathcal{D}(S_1) + \mathcal{D}(S_2)$ to be greater than ϵ, we must have $\mathcal{D}(S_1) > \epsilon/2$ or $\mathcal{D}(S_2) > \epsilon/2$. Therefore, by the union bound

$$\Pr\left[\mathcal{D}(\{x \mid h(x) \neq f(x)\}) > \epsilon\right] \leq \Pr\left[\mathcal{D}(S_1) > \epsilon/2\right] + \Pr\left[\mathcal{D}(S_2) > \epsilon/2\right] \tag{21}$$

Without loss of generality, let us focus on sector S_1. There are two other sectors that are relevant in analyzing S_1. The first one is the sector defined going from the location of f to the location of h in trigonometric order, we will refer to this sector as S_c. The second one is the error of h with respect to f', we will refer to this sector as S_e. Note that S_1, S_c, and S_e all have the same area.

If we assume that the density is non-negative, then there exists some positive ϵ' such that

$$\Pr[\mathcal{D}(S_1) \geq \epsilon/2] \leq \Pr[\mathcal{D}(S_c) \geq \epsilon'] \tag{22}$$

and likewise, there exists some positive ϵ'' such that

$$\Pr[\mathcal{D}(S_c) \geq \epsilon/2] \leq \Pr[\mathcal{D}(S_e) \geq \epsilon''] \tag{23}$$

Finally, we know from theorem 1 that for all ϵ, $\Pr[\mathcal{D}(S_1') > \epsilon] < (1-\epsilon)^m$. We can conclude that there exists some decreasing function a such that

$$\Pr[\mathcal{D}(S_1) \geq \epsilon/2] \leq a(m) \tag{24}$$

The same arguments holds for S_2 and since a is a decreasing function of m, for any value δ we can choose m large enough. □

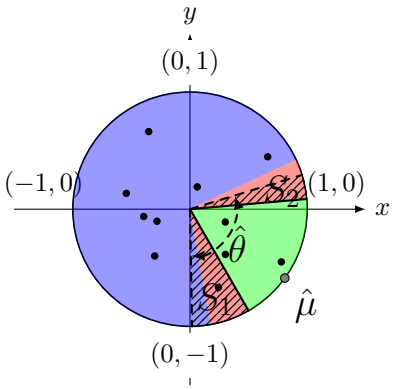

Figure 16: Error of A_{EFM} w.r.t. f'

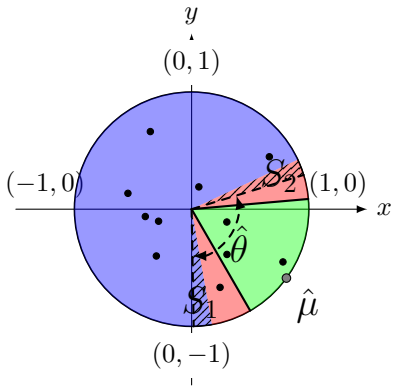

Figure 17: Error of A_{EFM} w.r.t. f

1. With respect to f, A_{EFM} is probably approximately correct for non-negative densities
2. With respect to f', A_{EFM} has an error lower bound of $\mathcal{D}(S_1) + \mathcal{D}(S_2)$