Supplemental Materials

A Proofs to theorems in the paper

We start by listing the basic inequalities that are frequently used in our proof.

Inequality 1.

$$\begin{aligned}\left\|\sum_{i=1}^{n}(x_i-y_i)\right\|_2^2 =&\sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle x_i-y_i, x_j-y_j\right\rangle\\ =&\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\left(\|x_i-y_i\|_2^2+\|x_j-y_j\|_2^2-\|(x_i-y_i)-(x_j-y_j)\|_2^2\right)\\ \leq& n\sum_{i=1}^{n}\|x_i-y_i\|_2^2 \end{aligned}\tag{21}$$

If

$$\mathbb{E}[x_i-y_i]=0, \forall i,$$

then

$$\mathbb{E}\left\|\sum_{i=1}^{n}(x_i-y_i)\right\|_2^2=\sum_{i=1}^{n}\mathbb{E}\|x_i-y_i\|_2^2+\sum_{i\neq j}^{n}\mathbb{E}\left\langle x_i-y_i, x_j-y_j\right\rangle=\sum_{i=1}^{n}\mathbb{E}\|x_i-y_i\|_2^2\tag{22}$$

Inequality 2. From the Lipschitzan gradient assumption in (3) and Inequality in (21), we can obtain:

$$\left\|\nabla f(x)-\nabla f(y)\right\|_2^2=\left\|\frac{1}{n}\sum_{i=1}^{n}\left(\nabla f_i(x)-\nabla f_i(y)\right)\right\|_2^2\leq\frac{1}{n}\sum_{i=1}^{n}\left\|\nabla f_i(x)-\nabla f_i(y)\right\|_2^2\leq L^2\|x-y\|_2^2\tag{23}$$

Thus,

$$\left\|\nabla f(x)-\nabla f(y)\right\|_2\leq L\|x-y\|_2\tag{24}$$

which is equivalent to

$$f(y)\leq f(x)+\nabla f(x)^T(y-x)+\frac{L}{2}\|y-x\|_2^2.\tag{25}$$

To see this, let us consider a function $g(x)=\frac{L}{2}x^Tx-f(x)$. Because $\nabla g(x)=Lx-\nabla f(x)$, we have $(x-y)\left(\nabla g(x)-\nabla g(y)\right)=L\|x-y\|_2^2-\left(\nabla f(x)-\nabla f(y)\right)^T(x-y)\geq 0$, which implies $g(x)$ is convex. Note that $f(x)$ can be non-convex. It follows that $g(y)\geq g(x)+\nabla g(x)^T(y-x)$, i.e., $\frac{L}{2}y^Ty-f(y)\geq\frac{L}{2}x^Tx-f(x)+\left(Lx-\nabla f(x)\right)^T(y-x)$, which is equivalent to (25).

Proof to Theorem 1. The computation of Algorithm 1 can be expressed as

$$X_{j+1}=\mathcal{P}_j(X_j-\gamma\partial F(X_j;\xi_j))W_n+(\mathcal{I}-\mathcal{P}_j)(X_j-\gamma\partial F(X_j;\xi_j))\tag{26}$$

Here, $X_j=[x_{j,1},x_{j,2},...,x_{j,n}]\in\mathbb{R}^{N\times n}$ are the model parameters on n nodes, $\mathcal{P}_j$ is the projection matrix representing selected components as defined in the algorithm, W_n is a n by n matrix with all elements being $1/n$, and $\partial F(X_j;\xi_j)=[\nabla F_1(x_{j,1};\xi_{j,1}),\nabla F_2(x_{j,2};\xi_{j,2}),...,\nabla F_n(x_{j,n};\xi_{j,n})]$ are the stochastic gradients. Our goal is to bound the average squared norm of gradients over K iterations, i.e.,

$$\frac{1}{K}\left(\sum_{j=0}^{K-1}\left\|\nabla f\left(\frac{X_j\mathbf{1_n}}{n}\right)\right\|_2^2\right).$$

From (26) and because $W_n\frac{\mathbf{1}_n}{n}=\frac{\mathbf{1}_n}{n}$, we have:

$$\begin{aligned}\mathbb{E}f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right)=&\mathbb{E}f\left(\mathcal{P}_j(X_j-\gamma\partial F(X_j;\xi_j))\frac{\mathbf{1}_n}{n}+(\mathcal{I}-\mathcal{P}_j)(X_j-\gamma\partial F(X_j;\xi_j))\frac{\mathbf{1}_n}{n}\right)\\ =&\mathbb{E}f\left(\left(X_j-\gamma\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n}\right)\end{aligned}\tag{27}$$

Then, we apply Lipschitzan gradient assumption in (25) to (27) and obtain:

$$\mathbb{E}f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right) \le \mathbb{E}f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \underbrace{\mathbb{E}\left\langle \nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right), \gamma\partial F\left(X_j;\xi_j\right)\frac{\mathbf{1}_n}{n}\right\rangle}_{=:T_1} + \frac{L}{2}\underbrace{\mathbb{E}\left\|\gamma\partial F(X_j,\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_2} \tag{28}$$

For T_1, we have:

$$\begin{aligned} T_1 :=&\mathbb{E}\left\langle \nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right), \gamma\partial F\left(X_j;\xi_j\right)\frac{\mathbf{1}_n}{n}\right\rangle \\ \overset{\because \text{ Assumption (4)}}{=}&\gamma\mathbb{E}\left\langle \nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right), \partial f\left(X_j\right)\frac{\mathbf{1}_n}{n}\right\rangle \\ =&\frac{\gamma}{2}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 + \frac{\gamma}{2}\mathbb{E}\left\|\partial f\left(X_j\right)\frac{\mathbf{1}_n}{n}\right\|_2^2 - \frac{\gamma}{2}\underbrace{\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_3} \end{aligned} \tag{29}$$

$$\begin{aligned} T_3 :=&\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 = \mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i\left(X_j\frac{\mathbf{1}_n}{n}\right) - \frac{1}{n}\sum_{i=1}^{n}\nabla f_i\left(X_j\mathbf{e}_i\right)\right\|_2^2 \\ \overset{\text{According to (21)}}{\le}&\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i\left(X_j\frac{\mathbf{1}_n}{n}\right) - \nabla f_i\left(X_j\mathbf{e}_i\right)\right\|_2^2 \\ \overset{\text{Assumption (3)}}{\le}&\frac{L^2}{n}\underbrace{\sum_{i=1}^{n}\mathbb{E}\left\|X_j\frac{\mathbf{1}_n}{n} - X_j\mathbf{e}_i\right\|_2^2}_{=:T_{4j}} \end{aligned} \tag{30}$$

If we plug T_3 and T_1 back to (28), we can get:

$$\begin{aligned} \mathbb{E}f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right) \le&\mathbb{E}f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \frac{\gamma}{2}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 - \frac{\gamma}{2}\mathbb{E}\left\|\partial f\left(X_j\right)\frac{\mathbf{1}_n}{n}\right\|_2^2 \\ &+ \frac{L^2\gamma}{2}\boxed{\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|X_j\frac{\mathbf{1}_n}{n} - X_j\mathbf{e}_i\right\|_2^2} + \frac{L\gamma^2}{2}\mathbb{E}\left\|\partial F(X_j,\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 \end{aligned} \tag{31}$$

Note that the term in the box is the variance of model parameters among nodes. We can see that the smaller the variance of model parameters among nodes, the smaller the bound of the squared norm of gradient at iteration j. Intuitively, a smaller variance of model parameters among nodes indicates that the trajectories of training on different nodes are not far apart, and thus, leads to better convergence of the algorithm. Therefore, the key to our proof is to bound the variance of model parameters (i.e., T_{4j}/n).

From (26), we can derive

$$\begin{aligned} X_j\left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i\right) =&(\mathcal{I} - \mathcal{P}_{j-1})(X_{j-1} - \gamma\partial F(X_{j-1};\xi_{j-1}))\left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i\right) \\ =&(\mathcal{I} - \mathcal{P}_{j-1})\left((\mathcal{I} - \mathcal{P}_{j-2})(X_{j-2} - \gamma\partial F(X_{j-2};\xi_{j-2}))\right)\left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i\right) \\ &- \gamma(\mathcal{I} - \mathcal{P}_{j-1})\partial F(X_{j-1};\xi_{j-1})\left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i\right) \\ =&\left(\prod_{t=0}^{j-1}(\mathcal{I} - \mathcal{P}_{j-1-t})X_0 - \gamma\sum_{k=0}^{j-1}\prod_{t=0}^{k}(\mathcal{I} - \mathcal{P}_{j-1-t})\partial F(X_{j-1-k};\xi_{j-1-k})\right)\left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i\right) \\ \overset{\because x_{0,i}=x_0,\forall i}{=}&\gamma\sum_{k=0}^{j-1}\prod_{t=0}^{k}(\mathcal{I} - \mathcal{P}_{j-1-t})\partial F(X_{j-1-k};\xi_{j-1-k})\left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i\right) \end{aligned} \tag{32}$$

Therefore, we have

$$\begin{aligned}
T_{4j} :=& \sum_{i=1}^{n} \mathbb{E}\left\| X_j \frac{\mathbf{1}_n}{n} - X_j \mathbf{e}_i \right\|_2^2 \\
=& \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{j-1} \prod_{t=0}^{k} (\mathcal{I} - \mathcal{P}_{j-1-t}) \partial F(X_{j-1-k}; \xi_{j-1-k}) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
=& \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{j-1} \prod_{t=0}^{k} (\mathcal{I} - \mathcal{P}_{j-1-t}) \left(\partial F(X_{j-1-k}; \xi_{j-1-k}) - \partial f(X_{j-1-k}) + \partial f(X_{j-1-k}) \right) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
=& \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{j-1} \prod_{t=0}^{k} (\mathcal{I} - \mathcal{P}_{j-1-t}) \left(\partial F(X_{j-1-k}; \xi_{j-1-k}) - \partial f(X_{j-1-k}) \right) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
& + \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{j-1} \prod_{t=0}^{k} (\mathcal{I} - \mathcal{P}_{j-1-t}) \partial f(X_{j-1-k}) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
\overset{\because \text{ Condition in (8)}}{\leq}& \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{p-1} \prod_{t=0}^{k} (\mathcal{I} - \mathcal{P}_{j-1-t}) \left(\partial F(X_{j-1-k}; \xi_{j-1-k}) - \partial f(X_{j-1-k}) \right) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
& + \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{p-1} \prod_{t=0}^{k} (\mathcal{I} - \mathcal{P}_{j-1-t}) \partial f(X_{j-1-k}) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
\leq& \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{p-1} \left(\partial F(X_{j-1-k}; \xi_{j-1-k}) - \partial f(X_{j-1-k}) \right) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
& + \gamma^2 \sum_{i=1}^{n} \mathbb{E}\left\| \sum_{k=0}^{p-1} \partial f(X_{j-1-k}) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
\overset{\text{According to (21) and (22)}}{\leq}& \gamma^2 \sum_{i=1}^{n} \sum_{k=0}^{p-1} \underbrace{\mathbb{E}\left\| \left(\partial F(X_{j-1-k}; \xi_{j-1-k}) - \partial f(X_{j-1-k}) \right) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2}_{=:T_5} \\
& + \gamma^2 p \sum_{i=1}^{n} \sum_{k=0}^{p-1} \underbrace{\mathbb{E}\left\| \partial f(X_{j-1-k}) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2}_{=:T_6}
\end{aligned} \tag{33}$$

$$\begin{aligned}
T_5 :=& \mathbb{E}\left\| \left(\partial F(X_{j-1-k}; \xi_{j-1-k}) - \partial f(X_{j-1-k}) \right) \left(\frac{\mathbf{1}_n}{n} - \mathbf{e}_i \right) \right\|_2^2 \\
\leq& \mathbb{E}\left\| \partial F(X_{j-1-k}; \xi_{j-1-k}) - \partial f(X_{j-1-k}) \right\|_F^2 \\
=& \sum_{i=1}^{n} \mathbb{E}\left\| \nabla F_i(x_{j-1-k,i}; \xi_{j-1-k,i}) - \nabla f_i(x_{j-1-k,i}) \right\|_2^2 \\
=& \sum_{i=1}^{n} \mathbb{E}\left\| \frac{1}{m} \sum_{s=1}^{m} \nabla F_i(x_{j-1-k,i}; \xi_{j-1-k,i,s}) - \nabla f_i(x_{j-1-k,i}) \right\|_2^2 \\
\overset{\text{According to (22)}}{=}& \frac{1}{m^2} \sum_{i=1}^{n} \sum_{s=1}^{m} \mathbb{E}\left\| \nabla F_i(x_{j-1-k,i}; \xi_{j-1-k,i,s}) - \nabla f_i(x_{j-1-k,i}) \right\|_2^2 \\
\leq& \frac{n\sigma^2}{m}
\end{aligned} \tag{34}$$

$$
\begin{aligned}
T_6 :=& \mathbb{E}\left\|\partial f(X_{j-1-k})\left(\frac{\mathbf{1}_n}{n}-\mathbf{e}_i\right)\right\|_2^2\\
\leq& \mathbb{E}\left\|\partial f(X_{j-1-k})\right\|_F^2 = \sum_{i=1}^{n}\mathbb{E}\|\nabla f_i(x_{j-1-k,i})\|_2^2\\
=& \sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i(x_{j-1-k,i}) - \nabla f(x_{j-1-k,i}) + \nabla f(x_{j-1-k,i}) - \nabla f\left(X_{j-1-k}\frac{\mathbf{1}_n}{n}\right) + \nabla f\left(X_{j-1-k}\frac{\mathbf{1}_n}{n}\right)\right\|_2^2\\
\overset{\text{According to (21)}}{\leq}& 3\sum_{i=1}^{n}\mathbb{E}\|\nabla f_i(x_{j-1-k,i}) - \nabla f(x_{j-1-k,i})\|_2^2 + 3\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f(x_{j-1-k,i}) - \nabla f\left(X_{j-1-k}\frac{\mathbf{1}_n}{n}\right)\right\|_2^2\\
&+3\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f\left(X_{j-1-k}\frac{\mathbf{1}_n}{n}\right)\right\|_2^2\\
\overset{\because\text{ Assumptions (6) and (3)}}{\leq}& 3n\varsigma^2 + 3L^2\underbrace{\sum_{i=1}^{n}\mathbb{E}\left\|x_{j-1-k,i} - X_{j-1-k}\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_{4j-1-k}} + 3n\mathbb{E}\left\|\nabla f\left(X_{j-1-k}\frac{\mathbf{1}_n}{n}\right)\right\|_2^2
\end{aligned}
\tag{35}
$$

From (33), (34) and (35), we have the bound for T_{4j}:

$$
T_{4j} \leq \frac{p\gamma^2 n^2\sigma^2}{m} + 3p^2\gamma^2 n^2\varsigma^2 + 3pnL^2\gamma^2\sum_{k=0}^{p-1}T_{4j-1-k} + 3pn^2\gamma^2\sum_{k=0}^{p-1}\mathbb{E}\left\|\nabla f\left(X_{j-1-k}\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 \tag{36}
$$

It follows that

$$
\sum_{j=0}^{K-1}T_{4j} \leq \frac{p\gamma^2 n^2\sigma^2 K}{m} + 3p^2\gamma^2 n^2\varsigma^2 K + 3np^2L^2\gamma^2\sum_{j=0}^{K-1}T_{4j} + 3p^2n^2\gamma^2\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 \tag{37}
$$

Suppose

$$
1 > 3np^2L^2\gamma^2 \tag{38}
$$

we have

$$
\sum_{j=0}^{K-1}T_{4j} \leq \frac{pn^2\gamma^2\left(K\left(\frac{\sigma^2}{m}+3p\varsigma^2\right)+3p\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2\right)}{1-3np^2L^2\gamma^2} \tag{39}
$$

For T_2, we have

$$
\begin{aligned}
T_2 :=& \mathbb{E}\left\|\gamma\partial F(X_j,\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=& \gamma^2\mathbb{E}\left\|\partial F(X_j,\xi_j)\frac{\mathbf{1}_n}{n} - \partial f(X_j)\frac{\mathbf{1}_n}{n} + \partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=& \gamma^2\mathbb{E}\left\|\partial F(X_j,\xi_j)\frac{\mathbf{1}_n}{n} - \partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 + \gamma^2\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=& \gamma^2\mathbb{E}\left\|\frac{1}{mn}\sum_{i=1}^{n}\sum_{s=1}^{m}\nabla F_i(x_{j,i},\xi_{j,i,s}) - \nabla f_i(x_{j,i})\right\|_2^2 + \gamma^2\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
\overset{\text{According to (22)}}{\leq}& \frac{\gamma^2\sigma^2}{mn} + \gamma^2\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2
\end{aligned}
\tag{40}
$$

From (27), (29), (30) and (40), we can obtain

$$
\mathbb{E}f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right) \leq \mathbb{E}f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \frac{\gamma}{2}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 - \frac{\gamma}{2}\mathbb{E}\left\|\partial f\left(X_j\right)\frac{\mathbf{1}_n}{n}\right\|_2^2 + \frac{L^2\gamma}{2n}T_{4j}
$$

$$+\frac{L\gamma^2\sigma^2}{2mn}+\frac{L\gamma^2}{2}\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 \tag{41}$$

Suppose

$$\frac{\gamma}{2}\geq\frac{L\gamma^2}{2} \tag{42}$$

we have

$$\mathbb{E}f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right)\leq\mathbb{E}f\left(X_j\frac{\mathbf{1}_n}{n}\right)-\frac{\gamma}{2}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2+\frac{L^2\gamma}{2n}T_{4j}+\frac{L\gamma^2\sigma^2}{2mn} \tag{43}$$

Summing up from $\mathbb{E}f(x_1)$ to $\mathbb{E}f(x_K)$, we have:

$$\mathbb{E}f(x_K)\leq\mathbb{E}f(x_0)-\frac{\gamma}{2}\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2+\frac{L^2\gamma}{2n}\sum_{j=0}^{K-1}T_{4j}+\frac{L\gamma^2\sigma^2K}{2mn} \tag{44}$$

Plugging (39) into (45), we can obtain

$$\begin{aligned}\mathbb{E}f(x_K)\leq&\mathbb{E}f(x_0)-\frac{\gamma}{2}\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2+\frac{L^2\gamma}{2n}\cdot\frac{pn^2\gamma^2K\left(\frac{\sigma^2}{m}+3p\varsigma^2\right)}{1-3np^2L^2\gamma^2}\\&+\frac{L^2\gamma}{2n}\cdot\frac{3p^2n^2\gamma^2}{1-3np^2L^2\gamma^2}\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2+\frac{L\gamma^2\sigma^2K}{2mn}\end{aligned} \tag{45}$$

Suppose

$$\frac{\gamma}{2}>\frac{L^2\gamma}{2n}\cdot\frac{3p^2n^2\gamma^2}{1-3np^2L^2\gamma^2} \tag{46}$$

i.e.,

$$1>6np^2L^2\gamma^2, \tag{47}$$

then it follows from (45) that

$$\frac{1}{K}\left(\sum_{j=0}^{K-1}\left\|\nabla f\left(\frac{X_j\mathbf{1_n}}{n}\right)\right\|_2^2\right)\leq\frac{2\left(f\left(x_0\right)-f^*\right)D_1}{\gamma K}+\frac{L\gamma\sigma^2D_1}{M}+\frac{pn^2L^2\gamma^2\sigma^2D_2}{M}+3np^2L^2\gamma^2\varsigma^2D_2, \tag{48}$$

where

$$D_1=\left(\frac{1-3np^2L^2\gamma^2}{1-6np^2L^2\gamma^2}\right),\quad D_2=\left(\frac{1}{1-6np^2L^2\gamma^2}\right). \tag{49}$$

The above bound holds when (38), (42) and (47) are satisfied, which is Theorem 1. □

If we set $\gamma=\theta\sqrt{M/K}$ and let $D_1\leq 2$ and $D_2\leq 2$, i.e.,

$$1\geq 12np^2L^2\gamma^2\iff K\geq 12nMp^2L^2\theta^2,$$

we can obtain Corollary 1 from Theorem 1.

Proof to Theorem 2. The computation of distributed SGD with gradient quantization can be expressed as

$$x_{j+1}=x_j-\gamma Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n} \tag{50}$$

Here, $X_j=[x_{j,1},x_{j,2},...,x_{j,n}]\in\mathbb{R}^{N\times n}$ are the model parameters on n nodes, Q is any unbiased stochastic quantization function, and $\partial F(X_j;\xi_j)=[\nabla F_1(x_{j,1};\xi_{j,1}),\nabla F_2(x_{j,2};\xi_{j,2}),...,\nabla F_n(x_{j,n};\xi_{j,n})]$ are the stochastic gradients. Note that in this algorithm $x_{j,i}=x_j,\forall j,\forall i$.

Consider $\mathbb{E}f(x_{+1})$:

$$\mathbb{E}f(x_{j+1})=\mathbb{E}f\left(x_j-\gamma Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n}\right)$$

$$\overset{\text{According to (25)}}{\leq} \mathbb{E} f(x_j) - \underbrace{\mathbb{E}\left\langle \nabla f(x_j), \gamma Q\left(\partial F(X_j;\xi_j)\right) \frac{\mathbf{1}_n}{n}\right\rangle}_{=:T_1} + \frac{L}{2}\underbrace{\mathbb{E}\left\|\gamma Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_2} \tag{51}$$

For T_1 and T_2, we have:

$$\begin{aligned}
T_1 :=& \mathbb{E}\left\langle \nabla f(x_j), \gamma Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n}\right\rangle \\
\overset{\text{Law of total expectation}}{=}& \gamma\mathbb{E}\left[\mathbb{E}_{Q\sim\mathcal{H}}\left\langle \nabla f(x_j), Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n}\right\rangle\right] \\
\overset{Q\text{ is unbiased}}{=}& \gamma\mathbb{E}\left\langle \nabla f(x_j), \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\rangle \\
=& \gamma\mathbb{E}\left[\mathbb{E}_{\xi_j\sim\mathcal{D}}\left\langle \nabla f(x_j), \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\rangle\right] \\
\overset{\text{Assumption (4)}}{=}& \gamma\mathbb{E}\left\langle \nabla f(x_j), \partial f(X_j)\frac{\mathbf{1}_n}{n}\right\rangle \\
\overset{x_{j,i}=x_j,\forall i}{=}& \gamma\mathbb{E}\left\langle \nabla f(x_j), \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_j)\right\rangle \overset{\text{According to (2)}}{=} \gamma\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2
\end{aligned} \tag{52}$$

$$\begin{aligned}
T_2 :=& \mathbb{E}\left\|\gamma Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n}\right\|_2^2 \\
=& \gamma^2\mathbb{E}\left\|Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n}\right\|_2^2 \\
=& \gamma^2\mathbb{E}\left\|Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n} - \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n} + \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 \\
=& \gamma^2\mathbb{E}\left\|Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n} - \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 + \gamma^2\mathbb{E}\left\|\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 \\
&+ \gamma^2\mathbb{E}\left\langle Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n} - \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}, \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\rangle \\
\overset{\because\, Q\text{ is unbiased}}{=}& \gamma^2\underbrace{\mathbb{E}\left\|Q\left(\partial F(X_j;\xi_j)\right)\frac{\mathbf{1}_n}{n} - \partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_3} + \gamma^2\underbrace{\mathbb{E}\left\|\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_4}
\end{aligned} \tag{53}$$

$$\begin{aligned}
T_3 =& \mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(Q\left(\nabla F_i(x_j;\xi_{j,i})\right) - \nabla F_i(x_j;\xi_{j,i})\Big)\right\|_2^2 \\
\overset{\text{According to (22)}}{=}& \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|Q\left(\nabla F_i(x_j;\xi_{j,i})\right) - \nabla F_i(x_j;\xi_{j,i})\right\|_2^2 \\
\overset{\text{According to (13)}}{\leq}& \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla F_i(x_j;\xi_{j,i})\right\|_2^2 = \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla F_i(x_j;\xi_{j,i}) - \nabla f_i(x_j) + \nabla f_i(x_j)\right\|_2^2 \\
=& \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla F_i(x_j;\xi_{j,i}) - \nabla f_i(x_j)\right\|_2^2 + \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i(x_j)\right\|_2^2 \\
=& \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\frac{1}{m}\sum_{s=1}^{m}\nabla F_i(x_j;\xi_{j,i,s}) - \nabla f_i(x_j)\right\|_2^2 + \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i(x_j)\right\|_2^2 \\
\overset{\text{Assumption in (5)}}{\leq}& \frac{q\sigma^2}{mn} + \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i(x_j) - \nabla f(x_j) + \nabla f(x_j)\right\|_2^2 \\
\overset{\because \nabla f(x)=\frac{1}{n}\sum_i \nabla f_i(x)}{=}& \frac{q\sigma^2}{mn} + \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i(x_j) - \nabla f(x_j)\right\|_2^2 + \frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\langle \nabla f_i(x_j)-\nabla f(x_j),\nabla f(x_j)\right\rangle\\
=&\frac{q\sigma^2}{mn}+\frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f_i(x_j)-\nabla f(x_j)\right\|_2^2+\frac{q}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2\\
\le&\frac{q\sigma^2}{mn}+\frac{q\varsigma^2}{n}+\frac{q}{n}\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
T_4=&\mathbb{E}\left\|\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}-\partial f(X_j)\frac{\mathbf{1}_n}{n}+\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=&\mathbb{E}\left\|\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}-\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2+\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=&\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^{n}\left(\nabla F_i(x_j;\xi_{j,i})-\nabla f_i(x_j)\right)\right\|_2^2+\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2\\
\le&\frac{\sigma^2}{mn}+\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2
\end{aligned}
\tag{55}
$$

Plugging T_1, T_2 into (51), we have:

$$
\begin{aligned}
\mathbb{E}f(x_{j+1})\le&\mathbb{E}f(x_j)-\gamma\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2+\frac{(1+q)L\gamma^2\sigma^2}{2mn}+\frac{qL\gamma^2\varsigma^2}{2n}+\frac{(q+n)L\gamma^2}{2n}\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2\\
=&\mathbb{E}f(x_j)-\left(\gamma-\frac{(q+n)L\gamma^2}{2n}\right)\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2+\frac{(1+q)L\gamma^2\sigma^2}{2mn}+\frac{qL\gamma^2\varsigma^2}{2n}
\end{aligned}
\tag{56}
$$

Summing up from $\mathbb{E}f(x_1)$ to $\mathbb{E}f(x_K)$, we have:

$$
\mathbb{E}f(x_K)\le\mathbb{E}f(x_0)-\left(\gamma-\frac{(q+n)L\gamma^2}{2n}\right)\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f(x_j)\right\|_2^2+\frac{(1+q)L\gamma^2\sigma^2K}{2mn}+\frac{qL\gamma^2\varsigma^2K}{2n}
\tag{57}
$$

Thus, we have Theorem 2 if $(1+\frac{q}{n})L\gamma<2$:

$$
\frac{1}{K}\left(\sum_{j=0}^{K-1}\left\|\nabla f\left(x_j\right)\right\|_2^2\right)\le\frac{(f(x_0)-f^*)D}{\gamma K}+\frac{(1+q)L\gamma\sigma^2D}{2M}+\frac{qL\gamma\varsigma^2D}{2n},
\tag{58}
$$

where

$$
D=\frac{2}{2-(1+\frac{q}{n})L\gamma}
\tag{59}
$$

$\square$

If we set $\gamma=\theta\sqrt{M/K}$ and let $(1+\frac{q}{n})L\gamma\le 1$ (i.e., $D\le 2$) in Theorem 2, we can obtain Corollary 3.

Proof to Theorem 3. The computation of Algorithm 2 can be expressed as

$$
X_{j+1}=\begin{cases}X_j-\gamma\partial F(X_j;\xi_j) & \text{, if } (j+1) \bmod p\neq 0\\ X_{j+1-p}+Q\left(\left(X_j-\gamma\partial F(X_j;\xi_j)\right)-X_{j+1-p}\right)W_n & \text{, if } (j+1) \bmod p=0\end{cases}
\tag{60}
$$

Here, $X_j=[x_{j,1},x_{j,2},...,x_{j,n}]\in\mathbb{R}^{N\times n}$ are the model parameters on n nodes, Q is any unbiased stochastic quantization function, W_n is a n by n matrix with all elements being $1/n$, and $\partial F(X_j;\xi_j)=[\nabla F_1(x_{j,1};\xi_{j,1}),\nabla F_2(x_{j,2};\xi_{j,2}),...,\nabla F_n(x_{j,n};\xi_{j,n})]$ are the stochastic gradients.

The above formula can be expressed with projection matrix as

$$
X_{j+1}=\mathcal{P}_j(\widehat{X_j-\gamma\partial F(X_j;\xi_j)})W_n+(\mathcal{I}-\mathcal{P}_j)(X_j-\gamma\partial F(X_j;\xi_j)),
\tag{61}
$$

where $\mathcal{P}_j=\mathcal{I}$ when $j+1$ divides p; otherwise $\mathcal{P}_j=\mathbf{0}$, and

$$
(\widehat{X_j-\gamma\partial F(X_j;\xi_j)})=X_{j+1-p}+Q\left(\left(X_j-\gamma\partial F(X_j;\xi_j)\right)-X_{j+1-p}\right)
\tag{62}
$$

when $(j+1) \bmod p = 0$. Note that X_j have identical columns when j divides p.

Consider $\mathbb{E} f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right)$:

$$\begin{aligned}
\mathbb{E} f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right) = & \mathbb{E} f\left(X_j\frac{\mathbf{1}_n}{n} - \left(\gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n} + \mathcal{P}_j\left((X_j - \widehat{\gamma\partial F(X_j;\xi_j)}) - (X_j - \gamma\partial F(X_j;\xi_j))\right)\frac{\mathbf{1}_n}{n}\right)\right)\\
=&\mathbb{E} f\left(X_j\frac{\mathbf{1}_n}{n} - \left(\gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n} + \mathcal{P}_j\left(Q\left((X_j - \gamma\partial F(X_j;\xi_j)) - X_{j+1-p}\right) - (X_j - \gamma\partial F(X_j;\xi_j) - X_{j+1-p})\right)\frac{\mathbf{1}_n}{n}\right)\right)\\
\leq&\mathbb{E} f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \mathbb{E}\left\langle \nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right), \gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\rangle\\
&-\mathbb{E}\left\langle \nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right), \mathcal{P}_j\left(Q\left((X_j - \gamma\partial F(X_j;\xi_j)) - X_{j+1-p}\right) - (X_j - \gamma\partial F(X_j;\xi_j) - X_{j+1-p})\right)\frac{\mathbf{1}_n}{n}\right\rangle\\
&+\frac{L}{2}\mathbb{E}\left\|\gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n} + \mathcal{P}_j\left(Q\left((X_j - \gamma\partial F(X_j;\xi_j)) - X_{j+1-p}\right) - (X_j - \gamma\partial F(X_j;\xi_j) - X_{j+1-p})\right)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=&\mathbb{E} f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \underbrace{\mathbb{E}\left\langle \nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right), \gamma\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\rangle}_{=:T_1}\\
&+\frac{L}{2}\underbrace{\mathbb{E}\left\|\gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n} + \mathcal{P}_j\left(Q\left((X_j - \gamma\partial F(X_j;\xi_j)) - X_{j+1-p}\right) - (X_j - \gamma\partial F(X_j;\xi_j) - X_{j+1-p})\right)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_2}
\end{aligned}\tag{63}$$

For T_1 we have:

$$T_1 = \frac{\gamma}{2}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 + \frac{\gamma}{2}\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 - \frac{\gamma}{2}\underbrace{\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_3}\tag{64}$$

$$T_3 \leq \frac{L^2}{n}\underbrace{\sum_{i=1}^{n}\mathbb{E}\left\|X_j\frac{\mathbf{1}_n}{n} - X_j\mathbf{e}_i\right\|_2^2}_{=:T_{4j}}\tag{65}$$

Following the same steps from (32) to (39), we can get the same bound for $\sum_{j=0}^{K-1} T_{4j}$:

$$\sum_{j=0}^{K-1} T_{4j} \leq \frac{pn^2\gamma^2\left(K\left(\frac{\sigma^2}{m} + 3p\varsigma^2\right) + 3p\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2\right)}{1 - 3np^2L^2\gamma^2}\tag{66}$$

For T_2, we have

$$\begin{aligned}
T_2 :=&\mathbb{E}\left\|\gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n} + \mathcal{P}_j\left(Q\left((X_j - \gamma\partial F(X_j;\xi_j)) - X_{j+1-p}\right) - (X_j - \gamma\partial F(X_j;\xi_j) - X_{j+1-p})\right)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=&\mathbb{E}\left\|\gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
&+\mathbb{E}\left\|\mathcal{P}_j\left(Q\left((X_j - \gamma\partial F(X_j;\xi_j)) - X_{j+1-p}\right) - (X_j - \gamma\partial F(X_j;\xi_j) - X_{j+1-p})\right)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
&+\mathbb{E}\left\langle \gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}, \mathcal{P}_j\left(Q\left((X_j - \gamma\partial F(X_j;\xi_j)) - X_{j+1-p}\right) - (X_j - \gamma\partial F(X_j;\xi_j) - X_{j+1-p})\right)\frac{\mathbf{1}_n}{n}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}\left\|\gamma\partial F(X_j;\xi_j)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
&+\underbrace{\mathbb{E}\left\|\mathcal{P}_j\left(Q\left(\left(X_j-\gamma\partial F(X_j;\xi_j)\right)-X_{j+1-p}\right)-\left(X_j-\gamma\partial F(X_j;\xi_j)-X_{j+1-p}\right)\right)\frac{\mathbf{1}_n}{n}\right\|_2^2}_{=:T_{5j}}\\
\overset{\text{According to (40)}}{\le}&\frac{\gamma^2\sigma^2}{mn}+\gamma^2\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2+T_{5j}
\end{aligned}
\tag{67}
$$

Because $\mathcal{P}_j=\mathbf{0}$ when $(j+1)\bmod p\neq 0$, we have $T_{5j}=0$ when $(j+1)\bmod p\neq 0$.

When $(j+1)\bmod p=0$,

$$
\begin{aligned}
T_{5j}=&\mathbb{E}\left\|\left(Q\left(\left(X_j-\gamma\partial F(X_j;\xi_j)\right)-X_{j+1-p}\right)-\left(X_j-\gamma\partial F(X_j;\xi_j)-X_{j+1-p}\right)\right)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=&\gamma^2\mathbb{E}\left\|\left(Q\left(\sum_{k=j+1-p}^{j}\partial F(X_k;\xi_k)\right)-\sum_{k=j+1-p}^{j}\partial F(X_k;\xi_k)\right)\frac{\mathbf{1}_n}{n}\right\|_2^2\\
=&\gamma^2\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^{n}\left(Q\left(\sum_{k=j+1-p}^{j}\nabla F_i(x_{k,i};\xi_{k,i})\right)-\sum_{k=j+1-p}^{j}\nabla F_i(X_{k,i};\xi_{k,i})\right)\right\|_2^2\\
=&\frac{\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|Q\left(\sum_{k=j+1-p}^{j}\nabla F_i(x_{k,i};\xi_{k,i})\right)-\sum_{k=j+1-p}^{j}\nabla F_i(x_{k,i};\xi_{k,i})\right\|_2^2\\
\le&\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\sum_{k=j+1-p}^{j}\nabla F_i(x_{k,i};\xi_{k,i})\right\|_2^2\\
=&\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\sum_{k=j+1-p}^{j}\left(\nabla F_i(x_{k,i};\xi_{k,i})-\nabla f_i(x_{k,i})+\nabla f_i(x_{k,i})\right)\right\|_2^2\\
=&\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\sum_{k=j+1-p}^{j}\left(\nabla F_i(x_{k,i};\xi_{k,i})-\nabla f_i(x_{k,i})\right)\right\|_2^2+\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\sum_{k=j+1-p}^{j}\nabla f_i(x_{k,i})\right\|_2^2\\
=&\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\sum_{k=j+1-p}^{j}\mathbb{E}\left\|\nabla F_i(x_{k,i};\xi_{k,i})-\nabla f_i(x_{k,i})\right\|_2^2+\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\sum_{k=j+1-p}^{j}\nabla f_i(x_{k,i})\right\|_2^2\\
=&\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\sum_{k=j+1-p}^{j}\mathbb{E}\left\|\frac{1}{m}\sum_{s=1}^{m}\nabla F_i(x_{k,i};\xi_{k,i,s})-\nabla f_i(x_{k,i})\right\|_2^2+\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\sum_{k=j+1-p}^{j}\nabla f_i(x_{k,i})\right\|_2^2\\
\le&\frac{qp\gamma^2\sigma^2}{mn}+\frac{q\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left\|\sum_{k=j+1-p}^{j}\nabla f_i(x_{k,i})\right\|_2^2\\
\le&\frac{qp\gamma^2\sigma^2}{mn}+\frac{qp\gamma^2}{n^2}\sum_{i=1}^{n}\sum_{k=j+1-p}^{j}\mathbb{E}\left\|\nabla f_i(x_{k,i})\right\|_2^2\\
\le&\frac{qp\gamma^2\sigma^2}{mn}+\frac{qp\gamma^2}{n^2}\sum_{i=1}^{n}\sum_{k=j+1-p}^{j}\mathbb{E}\left\|\nabla f_i(x_{k,i})-\nabla f(x_{k,i})+\nabla f(x_{k,i})-\nabla f(X_k\frac{\mathbf{1}_n}{n})+\nabla f(X_k\frac{\mathbf{1}_n}{n})\right\|_2^2\\
\le&\frac{qp\gamma^2\sigma^2}{mn}+\frac{3qp^2\gamma^2\varsigma^2}{n}+\frac{3qp\gamma^2}{n^2}\sum_{k=j+1-p}^{j}T_{4k}+\frac{3qp\gamma^2}{n}\sum_{k=j+1-p}^{j}\mathbb{E}\left\|\nabla f(X_k\frac{\mathbf{1}_n}{n})\right\|_2^2
\end{aligned}
\tag{68}
$$

Plugging T_1, T_2 into (63), we can obtain:

$$\begin{aligned}\mathbb{E}f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right) \leq & \mathbb{E}f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \frac{\gamma}{2}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 - \frac{\gamma}{2}\mathbb{E}\left\|\partial F(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 + \frac{L^2\gamma}{2n}T_{4j} \\ & + \frac{L\gamma^2\sigma^2}{2mn} + \frac{L\gamma^2}{2}\mathbb{E}\left\|\partial f(X_j)\frac{\mathbf{1}_n}{n}\right\|_2^2 + \frac{L}{2}T_{5j} \end{aligned} \tag{69}$$

Suppose

$$\frac{\gamma}{2} > \frac{L\gamma^2}{2}, \tag{70}$$

we have

$$\begin{aligned}\mathbb{E}f\left(X_{j+1}\frac{\mathbf{1}_n}{n}\right) \leq & \mathbb{E}f\left(X_j\frac{\mathbf{1}_n}{n}\right) - \frac{\gamma}{2}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 + \frac{L^2\gamma}{2n}T_{4j} \\ & + \frac{L\gamma^2\sigma^2}{2mn} + \frac{L}{2}T_{5j} \end{aligned} \tag{71}$$

Summing up from $\mathbb{E}f(x_1)$ to $\mathbb{E}f(x_K)$, because $T_{5j} = 0$ when $(j+1) \bmod p \neq 0$, we can obtain:

$$\begin{aligned}\mathbb{E}f\left(X_K\frac{\mathbf{1}_n}{n}\right) \leq & \mathbb{E}f\left(x_0\right) - \frac{\gamma}{2}\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 + \frac{L^2\gamma}{2n}\sum_{j=0}^{K-1}T_{4j} \\ & + \frac{L\gamma^2\sigma^2K}{2mn} + \frac{L}{2}\left(\frac{q\gamma^2\sigma^2K}{mn} + \frac{3qp\gamma^2\varsigma^2K}{n} + \frac{3qp\gamma^2}{n^2}\sum_{j=0}^{K-1}T_{4j} + \frac{3qp\gamma^2}{n}\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f(X_k\frac{\mathbf{1}_n}{n})\right\|_2^2\right) \\ = & \mathbb{E}f\left(x_0\right) - (\frac{\gamma}{2} - \frac{3Lqp\gamma^2}{2n})\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 \\ & + \left(\frac{L^2\gamma}{2n} + \frac{3Lqp\gamma^2}{2n^2}\right)\sum_{j=0}^{K-1}T_{4j} + \frac{(1+q)L\gamma^2\sigma^2K}{2mn} + \frac{3Lqp\gamma^2\varsigma^2K}{2n} \end{aligned} \tag{72}$$

Plugging (66) into (72), if $1 - \frac{3Lqp\gamma}{n} - \frac{3p^2\gamma^2(nL^2+3qpL\gamma)}{1-3np^2L^2\gamma^2} > 0$, we can obtain:

$$\begin{aligned}\frac{1}{K}\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 \leq & \frac{2\left(f(x_0) - f^*\right)D}{\gamma K} + \frac{(1+q)L\gamma\sigma^2D}{M} + \frac{3Lqp\gamma\varsigma^2D}{n} \\ & + \frac{(nL^2 + 3qpL\gamma)p\gamma^2\sigma^2D}{m(1-3np^2L^2\gamma^2)} + \frac{3(nL^2+3qpL\gamma)p^2\gamma^2\varsigma^2D}{1-3np^2L^2\gamma^2} \end{aligned} \tag{73}$$

where

$$D = \frac{1}{\left(1 - \frac{3Lqp\gamma}{n} - \frac{3p^2\gamma^2(nL^2+3qpL\gamma)}{1-3np^2L^2\gamma^2}\right)} \tag{74}$$

If we set $\gamma = \theta\sqrt{M/K}$ and let

$$\frac{3Lqp\gamma}{n} \leq \frac{1}{4} \tag{75}$$

and

$$\frac{3p^2\gamma^2(nL^2+3qpL\gamma)}{1-3np^2L^2\gamma^2} \leq \frac{3Lqp\gamma}{n}, \tag{76}$$

we can get $D \leq 2$. Thus,

$$\begin{aligned}\frac{1}{K}\sum_{j=0}^{K-1}\mathbb{E}\left\|\nabla f\left(X_j\frac{\mathbf{1}_n}{n}\right)\right\|_2^2 \leq & \frac{2\theta^{-1}\left(f(x_0) - f^*\right)}{\sqrt{MK}} + \frac{2(1+q)L\theta\sigma^2}{\sqrt{MK}} + \frac{6qpmL\theta\varsigma^2}{\sqrt{MK}} \\ & + \frac{2qL\gamma\sigma^2}{mn} + \frac{6qpL\gamma\varsigma^2}{n} \\ = & \frac{2\theta^{-1}\left(f(x_0) - f^*\right)}{\sqrt{MK}} + \frac{2(1+q)L\theta\sigma^2}{\sqrt{MK}} + \frac{6qpmL\theta\varsigma^2}{\sqrt{MK}} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{2qL\theta\sigma^2}{\sqrt{MK}} + \frac{6qpmL\theta\varsigma^2}{\sqrt{MK}} \\
=&\frac{2\theta^{-1}\left(f(x_0) - f^*\right)}{\sqrt{MK}} + \frac{2(1+2q)L\theta\sigma^2}{\sqrt{MK}} + \frac{12qpmL\theta\varsigma^2}{\sqrt{MK}} \qquad (77)
\end{aligned}
$$

The condition in (76) is equivalent to

$$\gamma = \theta\sqrt{\frac{M}{K}} \leq \frac{\sqrt{(n^2pL)^2 + 12np^2q^2(1+L^2)} - n^2pL}{6np^2q(1+L^2)} \qquad (78)$$

which simplifies to

$$K \geq \frac{M\theta^2\left(\sqrt{(n^2pL)^2 + 12np^2q^2(1+L^2)} + n^2pL\right)^2}{4q^2} \qquad (79)$$

We can see that if K is large enough, the conditions in (79), (75) and (70) can be satisfied and thus the bound in (77) is valid, which gives us Theorem 3.

$\square$

B Error bound of QSGD proposed by Alistarh *et al.* [5]

Alistarh *et al.* [5] propose a quantization function controlled by a quantization level s. A gradient component v_i is quantized to either l/s or $(l+1)/s$ based on a Bernoulli distribution defined as

$$\begin{cases} \mathbb{P}\{b_i = (l+1)/s\} = \frac{s|v_i|}{\|v\|_2} - l \\ \mathbb{P}\{b_i = l/s\} = 1 - \frac{s|v_i|}{\|v\|_2} + l \end{cases}$$

The quantized value of v_i is defined as $Q_s(v_i) = b_i \cdot sign(v_i) \cdot \|v\|_2$.

Because

$$\mathbb{E}\left[Q_s(v_t)\right] = \frac{l+1}{s} \cdot sign(v_i) \cdot \|v\|_2 \cdot \left(\frac{s|v_i|}{\|v\|_2} - l\right) + \frac{l}{s} \cdot sign(v_i) \cdot \|v\|_2 \cdot \left(1 - \frac{s|v_i|}{\|v\|_2} + l\right) = v_i,$$

Q_s is unbiased.

Alistarh *et al.* have shown the error bound q for Q_s is $\min(N/s^2, \sqrt{N}/s)$ [5], which suggest s should be set to $\sqrt{N}$ to achieve $q = 1$.

We now show a tighter bound of q is $\min(N/(4s^2), \sqrt{N}/s)$. Because

$$\left\|Q_s(v_i) - v_i\right\|_2^2 = \left(\frac{l+1}{s} \cdot \|v\|_2 - |v_i|\right)^2$$

with probability $\frac{s|v_i|}{\|v\|_2} - l$, and

$$\left\|Q_s(v_i) - v_i\right\|_2^2 = \left(\frac{l}{s} \cdot \|v\|_2 - |v_i|\right)^2$$

with probability $1 - \frac{s|v_i|}{\|v\|_2} + l$, we have

$$\begin{aligned}
\mathbb{E}\left\|Q_s(v_i) - v_i\right\|_2^2 &= \left(\frac{l+1}{s} \cdot \|v\|_2 - |v_i|\right)^2 \left(\frac{s|v_i|}{\|v\|_2} - l\right) + \left(\frac{l}{s} \cdot \|v\|_2 - |v_i|\right)^2 \left(1 - \frac{s|v_i|}{\|v\|_2} + l\right) \\
&= \left(\left(\frac{l+1}{s} \cdot \|v\|_2 - |v_i|\right)^2 - \left(\frac{l}{s} \cdot \|v\|_2 - |v_i|\right)^2\right) \left(\frac{s|v_i|}{\|v\|_2} - l\right) \\
&\quad + \left(\frac{l}{s} \cdot \|v\|_2 - |v_i|\right)^2 \\
&= \left(\frac{\|v\|_2}{s}\left(\frac{2l+1}{s} \cdot \|v\|_2 - 2|v_i|\right)\right) \left(\frac{s|v_i|}{\|v\|_2} - l\right) + \left(\frac{l}{s} \cdot \|v\|_2 - |v_i|\right)^2 \\
&= \frac{(2l+1)|v_i|\|v\|_2}{s} - v_i^2 - \frac{l^2+l}{s^2}\|v\|_2^2 \\
&= \left(|v_i| - \frac{l\|v\|_2}{s}\right)\left(\frac{(l+1)\|v\|_2}{s} - |v_i|\right) && (80) \\
&\overset{\because |v_i| \in \left[\frac{l\|v\|_2}{s}, \frac{(l+1)\|v\|_2}{s}\right)}{\le} \frac{\|v\|_2^2}{4s^2} && (81)
\end{aligned}$$

Therefore,

$$\mathbb{E}\left\|Q_s(v) - v\right\|_2^2 \le \frac{N\|v\|_2^2}{4s^2} \tag{82}$$

It follows that

$$\frac{\mathbb{E}\left\|Q_s(v) - v\right\|_2^2}{\|v\|_2^2} \le \frac{N}{4s^2} \tag{83}$$

From (80) we can also get:

$$\mathbb{E}\left\|Q_s(v_i) - v_i\right\|_2^2 = \left(|v_i| - \frac{l\|v\|_2}{s}\right)\left(\frac{(l+1)\|v\|_2}{s} - |v_i|\right) \overset{\because |v_i| \in \left[\frac{l\|v\|_2}{s}, \frac{(l+1)\|v\|_2}{s}\right)}{\le} \left(|v_i| - \frac{l\|v\|_2}{s}\right)\left(\frac{\|v\|_2}{s}\right)$$

$$= \left(1 - \frac{l\|v\|_2}{s|v_i|}\right)\left(\frac{\|v\|_2|v_i|}{s}\right) \le \frac{\|v\|_2|v_i|}{s} \tag{84}$$

Thus,

$$\mathbb{E}\big\|Q_s(v) - v\big\|_2^2 \le \frac{\|v\|_2\|v\|_1}{s} \overset{\because \|v\|_1 \le \sqrt{N}\|v\|_2}{\le} \frac{\sqrt{N}\|v\|_2^2}{s} \tag{85}$$

It follows that

$$\frac{\mathbb{E}\big\|Q_s(v) - v\big\|_2^2}{\|v\|_2^2} \le \frac{\sqrt{N}}{s} \tag{86}$$

Combining (83) and (86), we have

$$\frac{\mathbb{E}\big\|Q_s(v) - v\big\|_2^2}{\|v\|_2^2} \le \min\left(\frac{N}{4s^2}, \frac{\sqrt{N}}{s}\right) \tag{87}$$

This result suggest that $s = \frac{\sqrt{N}}{2}$ can achieve $q = 1$.

However, this number is not the actual quantization level of this function, because l has a smaller bound $\frac{\sqrt{N}\|v\|_\infty}{2\|v\|_2}$. To illustrate this point, we present an equivalent quantization function Q_a as follows:

For a gradient vector v, we first identify $\|v\|_\infty$ and divide it into $\frac{\sqrt{N}\|v\|_\infty}{2\|v\|_2}$ intervals of size $\phi = \frac{2\|v\|_2}{\sqrt{N}}$. The absolute value of other components will fall into one of the intervals. Suppose a component v_i is in interval $[l\phi, (l+1)\phi)$, its quantized value is either the upper bound or the lower bound based on a Bernoulli distribution defined as

$$\begin{cases} \mathbb{P}\{b_i = (l+1)\} = \frac{|v_i|}{\phi} - l \\ \mathbb{P}\{b_i = l\} = 1 - \frac{|v_i|}{\phi} + l \end{cases} \tag{88}$$

The quantized value of v_i is defined as $Q_a(v_i) = b_i \cdot sign(v_i) \cdot \phi$. Because

$$\big\|Q_a(v_i) - v_i\big\|_2^2 = \begin{cases} (|v_i| - l\phi)^2 & \text{, if } b_i = l \\ (|v_i| - (l+1)\phi)^2 & \text{, if } b_i = (l+1) \end{cases}$$

we have

$$\mathbb{E}\big\|Q_a(v_i) - v_i\big\|_2^2 = (|v_i| - l\phi)\phi - (|v_i| - l\phi)^2 \le \frac{\phi^2}{4}, \text{for } v_i \in [l\phi, (l+1)\phi)$$

which implies $\mathbb{E}\big\|Q_a(v) - v\big\|_2^2 \le \|v\|_2^2, \forall v \in \mathbb{R}^N$. Thus, Q_a ensures $q = 1$. It is not hard to see that Q_a is unbiased.

Q_a is actually equivalent to Q_s proposed by Alistarh *et al.* because both function use interval of size $\phi = \frac{2\|v\|_2}{\sqrt{N}}$. The presentation of Q_a shows that the number of intervals is actually $\frac{\sqrt{N}\|v\|_\infty}{2\|v\|_2}$, which means that the number of bits to represent the quantized value is $\left\lceil \log_2\left(\frac{\sqrt{N}\|v\|_\infty}{2\|v\|_2}\right)\right\rceil$.

C Error bound of TernGrad proposed by Wen *et al.* [46]

For a gradient vector v, they first identify the component with the largest absolute value (i.e., $\|v\|_\infty$). Then, the gradient components are quantized to either 1 or 0 based on a Bernoulli distribution defined as

$$\begin{cases} \mathbb{P}\{b_i = 1\} = |v_i| / \|v\|_\infty \\ \mathbb{P}\{b_i = 0\} = 1 - |v_i| / \|v\|_\infty \end{cases}$$

The quantized value of v_i is defined as $Q_t(v_i) = b_i \cdot sign(v_i) \cdot \|v\|_\infty$. Because $\mathbb{E}\left[Q_t(v_i)\right] = sign(v_i) \cdot \|v\|_\infty \cdot |v_i| / \|v\|_\infty + 0 \cdot \left(1 - |v_i| / \|v\|_\infty\right) = v_i$, Q_t is unbiased.

We now consider its error bound. Because

$$\left\|Q_t(v_i) - v_i\right\|_2^2 = \begin{cases} v_i^2 & \text{, if } b_i = 0 \\ (\|v\|_\infty - |v_i|)^2 & \text{, if } b_i = 1 \end{cases}$$

we have

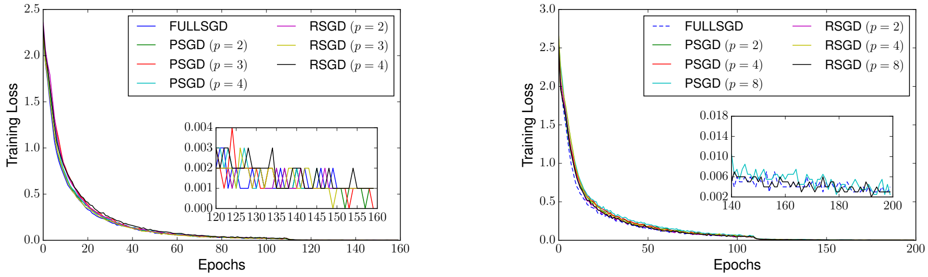
$$\mathbb{E}\left\|Q_t(v_i) - v_i\right\|_2^2 = \|v\|_\infty |v_i| - v_i^2$$

and thus,

$$\mathbb{E}\left\|Q_t(v) - v\right\|_2^2 = \|v\|_\infty \|v\|_1 - \|v\|_2^2$$

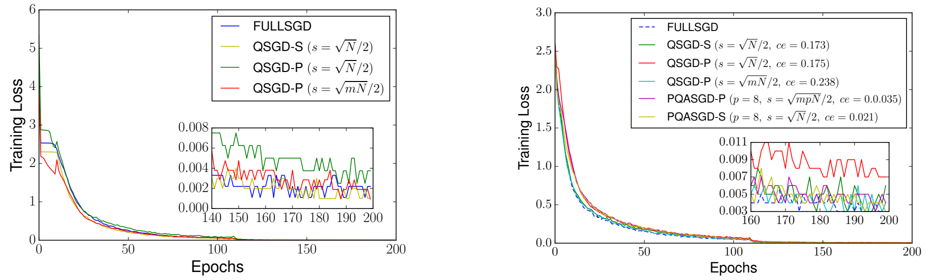
Since $\|v\|_\infty \le \|v\|_2 \le \|v\|_1 \le \sqrt{N}\|v\|_2, \forall v \in \mathbb{R}^N$, we have $\mathbb{E}\left\|Q_t(v) - v\right\|_2^2 \le (\sqrt{N} - 1)\|v\|_2^2$. Therefore, $q = \sqrt{N} - 1$, and `TernGrad` cannot achieve $O(1/\sqrt{MK})$ convergence rate in general.

D More experimental results on image classification



(a) VGG19 on 4 machines ($m = 32$, $\gamma = 0.1$ for epoch $0 - 109$, $\gamma = 0.01$ for epoch $110 - 159$)

(b) ResNet50 on 8 machines ($m = 32$, $\gamma = 0.2$ for epoch $0 - 109$, $\gamma = 0.02$ for epoch $110 - 199$)



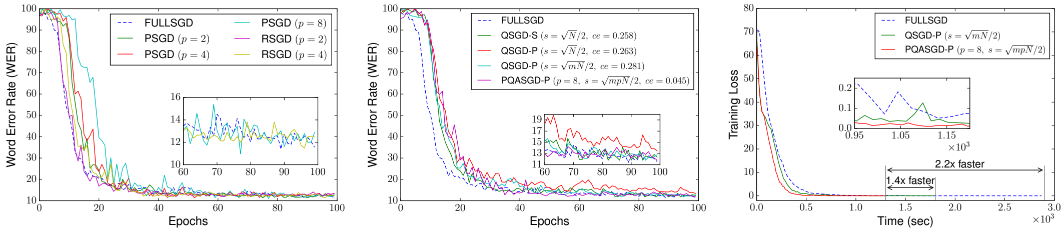
(c) ResNet110 on 8 machines ($m = 32$, $\gamma = 0.01$ for epoch $0 - 9$, $\gamma = 0.2$ for epoch $10 - 109$, $\gamma = 0.02$ for epoch $110 - 199$)

(d) ResNet50 on 8 machines ($m = 32$, $\gamma = 0.2$ for epoch $0 - 109$, $\gamma = 0.02$ for epoch $110 - 199$)

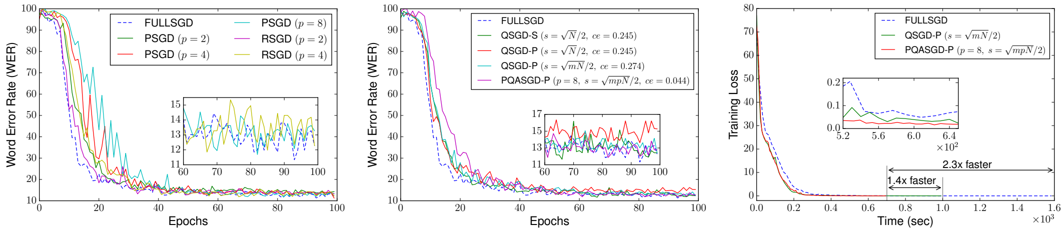
Figure 2: More results on image classification

Figure 2a and 2b validate that sparse parameter averaging can achieve $\mathcal{O}(1/\sqrt{MK})$ convergence rate. Figure 2c and 2d validate that 1) `QSGD` can achieve $\mathcal{O}(1/\sqrt{MK})$ convergence rate if configured properly, 2) our `PQASGD` can achieve $\mathcal{O}(1/\sqrt{MK})$ convergence rate with even less communication overhead, and 3) partitioned training data does have a negative effect to the convergence rate of distributed SGD with quantized gradients.

E Results on speech recognition



(a) 8 GPUs (b) 8 GPUs (c) 8 GPUs



(d) 16 GPUs (e) 16 GPUs (f) 16 GPUs

Figure 3: WER and training loss for a 5-layer LSTM on AN4.

We train a 5-layer LSTM of 800 hidden units per layer on AN4 dataset. The total mini-batch size M is set to 128. The learning rate is initialized to $3e^{-4}$ and decays in each epoch with annealing rate 1.01 (the learning rate is constant over iterations within an epoch).

Figure 3a compares the word error rate of `FULLSGD` with `PSGD` and `RSGD` with different settings of p on partitioned training data. We can see that, as the learning rate decreases, `PSGD` and `RSGD` match `FULLSGD` after 50 epochs. This validates our claim in Theorem 1 that sparse parameter averaging can achieve $\mathcal{O}(1/\sqrt{MK})$ convergence rate if K is large enough.

Figure 3b compares the word error rate of `FULLSGD` with different versions of `QSGD` and our `PQASGD`. We can observe a clear gap on the zoomed figure between `FULLSGD` and `QSGD-P` with s set to $\sqrt{N}/2$. This validates our claim that data partitioning affects the convergence rate of distributed SGD with gradient quantization. This effect is eliminated by setting s to $\sqrt{mN}/2$ for `QSGD-P` as we can see that all the other versions have similar WER after 50 epochs. Though our `PQASGD` converges slower than other versions in the first 40 epochs, it matches the WER of `FULLSGD` after 50 epochs. This validates our claim that our `PQASGD` converges at rate $\mathcal{O}(1/\sqrt{MK})$. Our `PQASGD` requires only 4.5% communication data size compared with `FULLSGD` in this case.

We then remove the validation at the end of each epoch and measure the execution time for training. Figure 3c shows the training loss over time for `FULLSGD`, `QSGD` and `PQASGD` on partitioned training data. We can see that our `PQASGD` runs 1.4x faster than `QSGD` and 2.2x faster than `FULLSGD`.

As shown in Figure 3d to 3f, the results on 16 GPUs follow a similar pattern.