Supplementary Material for Countering Feedback Delays in Multi-Agent Learning

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Abstract

Some of the missing proofs in the paper can be found here. All the results are stated again here (although the numberings are different from the main paper).

1 Examples of λ-Variationally Stable Games

Here we give two important classes of games that satisfy the λ-variational stability criterion. This is by no means a comprehensive list.

1. **Convex Potential Games** A game $\mathcal{G} = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{N} \mathcal{X}_i, \{u_i\}_{i=1}^{N})$ is called a potential game [1] if there exists a potential function $V : \mathcal{X} \to \mathbf{R}$ such that $u_i(x_i, \mathbf{x}_{-i}) - u_i(\tilde{x}_i, \mathbf{x}_{-i}) = V(x_i, \mathbf{x}_{-i}) - V_i(\tilde{x}_i, \mathbf{x}_{-i}), \forall i \in \mathcal{N}, \forall \mathbf{x} \in \mathcal{X}, \forall \tilde{x}_i \in \mathcal{X}_i$. A potential game is called a convex potential game if the potential function $V(\cdot)$ is concave[1] Note that in a convex potential game, we have

$$H_{ij}^{\lambda}(\mathbf{x}) = \frac{1}{2}\lambda_i \nabla_{x_j} v_i(\mathbf{x}) + \frac{1}{2}\lambda_j (\nabla_{x_i} v_j(\mathbf{x}))^T \tag{1.1}$$

$$= \frac{1}{2}\lambda_i \nabla_{x_j} \nabla_{x_i} V(\mathbf{x}) + \frac{1}{2}\lambda_j (\nabla_{x_i} \nabla_{x_j} V(\mathbf{x}))^T. \tag{1.2}$$

Setting $\lambda = \mathbf{1}$, we obtain $H^{\mathbf{1}}(\mathbf{x}) = \nabla^2 V$, which is negative semi-definite when V is concave. This implies that in a convex potential game, $\mathcal{C} = \arg\max_{\mathbf{x} \in \mathcal{X}} V(\mathbf{x})$ is $\mathbf{1}$-variationally stable per Lemma 2.2.

2. **Linear Cournot Oligopoly Games** There is a set $\mathcal{N} = \{1, 2, \dots, N\}$ of firms that supply the market with the same good (or service). Firm i provides $x_i \in [0, C_i]$ quantity to the market. The price of the good is a decreasing function of the total quantity of the good supplied to the market: $P(\mathbf{x}) = P(\sum_{i=1}^{N} x_i)$. A common price function takes the linear form: $P(\sum_{i=1}^{N} x_i) = a - b(\sum_{i=1}^{N} x_i), a > 0, b > 0$. The utility function for firm i is then $u_i(\mathbf{x}) = x_i P(\sum_{i=1}^{N} x_i) - c_i x_i$, where c_i is the unit production cost for firm i. In this case, one can clearly that this is a concave game. Further, again set setting $\lambda = \mathbf{1}$, we have

$$H_{ij}^{\mathbf{1}}(\mathbf{x}) = \frac{1}{2}\frac{\partial v_i(\mathbf{x})}{\partial x_j} + \frac{1}{2}\frac{\partial v_j(\mathbf{x})}{\partial x_i} \tag{1.3}$$

$$= -b\delta_{ij} - b, \tag{1.4}$$

[1] It is called convex potential game as opposed to concave potential game because in engineering, the utility is typically framed in terms of costs and convex costs correspond to concave utilities.

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where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Consequently,

$$H_{ij}^{\mathbf{1}}(\mathbf{x}) = -b(\mathbf{I} + \mathbf{1}_{N\times N}),$$

which is negative definite. Hence, the game admits a unique Nash equilibrium that is $\mathbf{1}$-variationally stable per Lemma 2.2.

2 λ-Variational Stability: A Key Criterion

Before proceeding, a word on the notation for the remainder of the supplementary material: for convenience, we shall write $v_j(\mathbf{x})(x_j - x_j^*)$ to denote the inner product between $v_j(\mathbf{x})$ and $x_j - x_j^*$ in replacement of the more cumbersome notation $\langle v_j(\mathbf{x}), x_j - x_j^*\rangle$.

Lemma 2.1. *If $\mathcal{C}$ is nonempty and λ-stable, then it is closed, convex and contains all Nash equilibria of the game.*

Proof. First we show that any element $\mathbf{x}^* \in \mathcal{C}$ is a Nash equilibrium. For any $i \in \mathcal{N}$, take any $x_i \in \mathcal{X}_i$ and any $\tau \in (0, 1]$, set $\mathbf{x} \triangleq (x_1^*, \ldots, x_{i-1}^*, (1-\tau)x_i^* + \tau x_i, x_{i+1}^*, \ldots, x_N^*) = \mathbf{x}^* + \tau(x_i - x_i^*)\mathbf{e}_i$, where $\mathbf{e}_i$ is the j-th unit vector in the standard basis. By convexity of $\mathcal{X}_i$, $\mathbf{x} \in \mathcal{X}$. Further,

$$\frac{d}{d\tau} u_i(x_i^* + \tau(x_i - x_i^*); \mathbf{x}_{-i}) = v_i(\mathbf{x})(x_i - x_i^*). \tag{2.1}$$

By applying the variational stability condition to the profiles $\mathbf{x}^*$ and $\mathbf{x}$, it follows that the RHS of the above equation is strictly negative for all $\tau > 0$. In turn, this implies that $u_i(\mathbf{x}) \le u_i(\mathbf{x}^*)$, i.e. $\mathbf{x}^*$ is a Nash equilibrium.

Next, we show that $\mathcal{C}$ is closed. Take any convergent sequence $\{\mathbf{x}^j\}_{j=0}^{\infty}$ in $\mathcal{C}$: $\mathbf{x}^j \in \mathcal{C}, \lim_{j\to\infty} \mathbf{x}^j = \mathbf{x}^*$. Then, for any $\mathbf{x} \in \mathcal{X}$, we have $\sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - x_i^j) \le 0, \forall j = 0, 1, \ldots$. Therefore, by continuity, it follows that $\lim_{j\to\infty} \sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - x_i^j) = \sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - x_i^*) \le 0, \forall \mathbf{x} \in \mathcal{C}$, thereby implying $\mathbf{x}^* \in \mathcal{C}$. Since $\{\mathbf{x}^j\}_{j=0}^{\infty}$ is any sequence in $\mathcal{C}$, $\mathcal{C}$ contains all its limit points and is therefore closed.

To see that $\mathcal{C}$ is convex, take any $\mathbf{x}^*, \mathbf{y}^* \in \mathcal{C}$ and any $\tau \in [0, 1]$. For any $\mathbf{x} \in \mathcal{X}$, we have

$$\sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - (\tau x_i^* - (1-\tau)y_i^*)) = \tag{2.2}$$

$$\tau \sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - x_i^*) + (1-\tau) \sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - y_i^*) \le 0, \tag{2.3}$$

thereby establishing that $\tau\mathbf{x}^* + (1-\tau)\mathbf{y}^* \in \mathcal{C}$.

Finally, to see that $\mathcal{C}$ contains all Nash equilibria of the game, assume that $\mathbf{z}^* \notin \mathcal{C}$ is a Nash equilibrium. We then have:

$$\sum_{i=1}^N \lambda_i v_i(\mathbf{z}^*)(x_i - z_i^*) \le 0, \forall \mathbf{x} \in \mathcal{X}. \tag{2.4}$$

Take an arbitrary $\mathbf{x}^* \in \mathcal{C}$. Since $\mathcal{C}$ is λ-variational stable and $\mathbf{z}^* \notin \mathcal{C}$, we have $\sum_{i=1}^N \lambda_i v_i(\mathbf{z}^*)(z_i^* - x_i^*) < 0$, implying that $\sum_{i=1}^N \lambda_i v_i(\mathbf{z}^*)(x_i^* - z_i^*) > 0$, which contradicts Equation 2.4. □

Lemma 2.2. *Consider a game with continuous actions $(\mathcal{N}, \mathcal{X} = \prod_{i=1}^N \mathcal{X}_i, \{u_i\}_{i=1}^N)$, where each u_i is twice continuously differentiable. For each $\mathbf{x} \in \mathcal{X}$, define the λ-weighted Hessian matrix $H^\lambda(\mathbf{x})$ as follows:*

$$H_{ij}^{\lambda}(\mathbf{x}) = \frac{1}{2}\lambda_i \nabla_{x_j} v_i(\mathbf{x}) + \frac{1}{2}\lambda_j (\nabla_{x_i} v_j(\mathbf{x}))^T. \tag{2.5}$$

If $H^\lambda(\mathbf{x})$ is negative-definite for every $\mathbf{x} \in \mathcal{X}$, then the game admits a unique Nash equilibrium $\mathbf{x}^*$ that is λ-globally variational stable.

Proof. Per Thereom 6 of [3], it follows that

$$\sum_{i=1}^{N} \lambda_i (v_i(\mathbf{x}) - v_i(\tilde{\mathbf{x}}))(x_i - \tilde{x}_i) \leq 0, \forall \mathbf{x}, \tilde{\mathbf{x}} \in \mathcal{X}, \tag{2.6}$$

where equality holds if and only if $\mathbf{x} = \tilde{\mathbf{x}}$. Per Theorem 2 of [3], this inequality then implies that there exists a unique Nash equilibrium $\mathbf{x}^*$. Plug $\mathbf{x}^*$ into Inequality 2.6 for $\tilde{\mathbf{x}}$, we have that for any $\mathbf{x} \in \mathcal{X}$:

$$\sum_{i=1}^{N} \lambda_i v_i(\mathbf{x})(x_i - x_i^*) \leq \sum_{i=1}^{N} \lambda_i v_i(\mathbf{x}^*)(x_i - x_i^*) \leq 0,$$

where the second inequality follows from $\mathbf{x}^*$ is a Nash equilibrium. Furthermore, both equality are achieved if and only if $\mathbf{x} = \mathbf{x}^*$. This implies that $\{\mathbf{x}^*\}$ is λ-variational stable. $\square$

3 Convergence under Synchronous and Bounded Delays

Algorithm 1 Online Mirror Descent on Games under Delays

Each player i chooses an initial y_i^0.
for $t = 0, 1, 2, \ldots$ **do**
  for $i = 1, \ldots, N$ **do**
    $x_i^t = \arg\max_{x_i \in \mathcal{X}_i} \{< y_i^t, x_i > - h_i(x_i)\}$
    $y_i^{t+1} = y_i^t + \alpha^t \sum_{s \in \mathcal{G}_i^t} v_i(\mathbf{x}^s)$
  end for
end for

Assumption 1. The delays are assumed to be:

1. Synchronous: $\mathcal{G}_i^t = \mathcal{G}_j^t, \forall i, j, \forall t$.

2. Bounded: $d_i^t \leq D, \forall i, \forall t$ (for some positive integer D).

Lemma 3.1. *For each $i \in \{1, \ldots, N\}$, let $h_i : \mathcal{X}_i \to \mathbb{R}$ be a regularizer with respect to the norm $\|\cdot\|_i$ that is K_i-strongly convex and let $\lambda \in \mathbb{R}_{++}^N$. Then $\forall \mathbf{x} \in \mathcal{X}, \forall \tilde{\mathbf{y}}, \mathbf{y} \in \mathbb{R}^{\sum_{i=1}^N d_i}$:*

1. $F^\lambda(\mathbf{x}, \mathbf{y}) \geq \frac{1}{2} \sum_{i=1}^N K_i \lambda_i \|C_i(y_i) - x_i\|_i^2 \geq \frac{1}{2}(\min_i K_i \lambda_i) \sum_{i=1}^N \|C_i(y_i) - x_i\|_i^2$.

2. $F^\lambda(\mathbf{x}, \tilde{\mathbf{y}}) \leq F^\lambda(\mathbf{x}, \mathbf{y}) + \sum_{i=1}^N \lambda_i \langle \tilde{y}_i - y_i, C_i(y_i) - x_i \rangle + \frac{1}{2}(\max_i \frac{\lambda_i}{K_i}) \sum_{i=1}^N (\|\tilde{y}_i - y_i\|_i^*)^2$, *where $\|\cdot\|_i^*$ is the dual norm of $\|\cdot\|_i$ (i.e. $\|y_i\|_i^* = \max_{\|x_i\|_i \leq 1} \langle x_i, y_i \rangle$.*

Theorem 3.2. *Consider a game with continuous actions $(\mathcal{N}, \mathcal{X} = \prod_{i=1}^N \mathcal{X}_i, \{u_i\}_{i=1}^N)$ that admits $\mathbf{x}^*$ as the unique Nash equilibrium that is λ-variationally stable. Under Assumption 1 for the delays, the OMD iterate $\mathbf{x}^t$ given in Algorithm 1 converges to $\mathbf{x}^*$, irrespective of the initial point $\mathbf{x}^0$.*

Remark 3.1. We repeat the four main steps in the following remark and prove each of them in detail in order.

1. Since the delays are synchronous, we denote by $\mathcal{G}^t$ the common set and d^t the common delay at round t. The gradient update in OMD under delays can then be written as:

$$y_i^{t+1} = y_i^t + \alpha^t \sum_{s \in \mathcal{G}^t} v_i(\mathbf{x}^s) = y_i^t + \alpha^t \left\{ |\mathcal{G}^t| v_i(\mathbf{x}^t) + \sum_{s \in \mathcal{G}^t} \{v_i(\mathbf{x}^s) - v_i(\mathbf{x}^t)\} \right\}. \tag{3.1}$$

   Define $b_i^t = \sum_{s \in \mathcal{G}^t} \{v_i(\mathbf{x}^s) - v_i(\mathbf{x}^t)\}$. We show that under the delay assumption (Assumption 1), $\lim_{t \to \infty} \|b_i^t\|_i^* = 0$ for each player i.

2. Define $\mathbf{b}^t = (b_1^t, \ldots, b_N^t)$ and we have $\lim_{t\to\infty} \mathbf{b}^t = 0$ per Claim 1. Since each player's gradient update can be written as $y_i^{t+1} = y_i^t + \alpha^t(|\mathcal{G}^t|v_i(\mathbf{x}^t) + b_i^t)$ per Claim 1, we can then write the joint OMD update (of all players) as:

$$\mathbf{x}^t = C(\mathbf{y}^t), \tag{3.2}$$
$$\mathbf{y}^{t+1} = \mathbf{y}^t + \alpha^t \left\{|\mathcal{G}^t|v(\mathbf{x}^t) + \mathbf{b}^t\right\}. \tag{3.3}$$

   Let $B(\mathbf{x}^*, \epsilon) \triangleq \{\mathbf{x} \in \mathcal{X} \mid \|\mathbf{x} - \mathbf{x}^*\| < \epsilon\}$ be the open ball centered around $\mathbf{x}^*$ with radius ϵ. Then, using λ-Fenchel coupling as a "energy" function and leveraging the handle on $\mathbf{b}^t$ given by Claim 1, we can establish that, for any $\epsilon > 0$ the iterate $\mathbf{x}^t$ will eventually enter $B(\mathbf{x}^*, \epsilon)$ and visit $B(\mathbf{x}^*, \epsilon)$ infinitely often, no matter what the initial point $\mathbf{x}^0$ is. Mathematically, the claim is that $\forall \epsilon > 0, \forall \mathbf{x}^0, |\{t \mid \mathbf{x}^t \in B(\mathbf{x}^*, \epsilon)\}| = \infty$.

3. Fix any $\delta > 0$ and consider the set $\tilde{B}(\mathbf{x}^*, \delta) \triangleq \{C(\mathbf{y}) \mid F^\lambda(\mathbf{x}^*, \mathbf{y}) < \delta\}$. In other words, $\tilde{B}(\mathbf{x}^*, \delta)$ is some "neighborhood" of $\mathbf{x}^*$, which contains every $\mathbf{x}$ that is an image of some $\mathbf{y}$ (under the choice map $C(\cdot)$) that is within δ distance of $\mathbf{x}^*$ under the λ-Fenchel coupling "metric". Although $F^\lambda(\mathbf{x}^*, \mathbf{y})$ is not a metric, $\tilde{B}(\mathbf{x}^*, \delta)$ contains an open ball within it. Mathematically, the claim is that for any $\delta > 0$, $\exists \epsilon(\delta) > 0$ such that: $B(\mathbf{x}^*, \epsilon) \subset \tilde{B}(\mathbf{x}^*, \delta)$.

4. For any "neighborhood" $\tilde{B}(\mathbf{x}^*, \delta)$, after long enough iterations, if $\mathbf{x}^t$ ever enters $\tilde{B}(\mathbf{x}^*, \delta)$, it will be trapped inside $\tilde{B}(\mathbf{x}^*, \delta)$ thereafter. Mathematically, the claim is that for any $\delta > 0$, there exists a $T(\delta)$, such that for any $t \geq T(\delta)$, if $\mathbf{x}^t \in \tilde{B}(\mathbf{x}^*, \delta)$, then $\mathbf{x}^{\tilde{t}} \in \tilde{B}(\mathbf{x}^*, \delta), \forall \tilde{t} \geq t$.

Putting all four elements above together, we note that the significance of Claim 3 is that, since the iterate $\mathbf{x}^t$ will enter $B(\mathbf{x}^*, \epsilon)$ infinitely often (per Claim 2), $\mathbf{x}^t$ must enter $\tilde{B}(\mathbf{x}^*, \delta)$ infinitely often. It therefore follows that, per Claim 4, starting from iteration t, $\mathbf{x}^t$ will remain in $\tilde{B}(\mathbf{x}^*, \delta)$. Since this is true for any $\delta > 0$, we have $F^\lambda(\mathbf{x}^*, \mathbf{y}^t) \to 0$ as $t \to \infty$. Per Statement 1 in Lemma 3.1, this leads to that $\|C(\mathbf{y}^t) - \mathbf{x}^*\| \to 0$ as $t \to \infty$, thereby establishing that $\mathbf{x}^t = C(\mathbf{y}^t) \to \mathbf{x}^*$ as $t \to 0$.

Proof:

1. We start by fixing some notation. Let $\mathbf{y}^t = (y_1^t, \ldots, y_N^t), \mathbf{x}^t = (x_1^t, \ldots, x_N^t)$ be the iterates generated in Algorithm 1. Since $\mathcal{X}$ is compact and $v(\cdot)$ is continuous, $V_{\max}^i \triangleq \max_{x_i \in \mathcal{X}_i} \|v_i(\mathbf{x})\|_i^* < \infty, V_{\max} \triangleq \max_{\mathbf{x} \in \mathcal{X}} \|v(\mathbf{x})\|^* = \sum_{i=1}^N V_{\max}^i < \infty$. Since each $h_i(\cdot)$ is K_i strongly convex (with respect to $\|\cdot\|_i$), it follows from a well-known result in convex analysis [2] that the choice map $C(\cdot)$ is $\frac{1}{K}$-Lipschitz continuous, where $K \triangleq \min_i K_i$. Finally, since each $v_i(\cdot)$ is Lipschitz continuous, $v(\cdot)$ is Lipschitz continuous as well and denote the Lipschitz constant as L.

   Since $d^t \leq D, \forall t$, it follows that $|\mathcal{G}^t| \leq D$ and $\min \mathcal{G}^t \geq t - D + 1$, for otherwise at least one gradient comes from $D + 1$ rounds before. Further, per the OMD update (first equality in Equation 3.1), we have:

$$\|\mathbf{y}^{t+1} - \mathbf{y}^t\|^* = \sum_{i=1}^N \|y_i^{t+1} - y_i^t\|_i^* = \sum_{i=1}^N \|\alpha^t \sum_{s \in \mathcal{G}^t} v_i(\mathbf{x}^s)\|_i^* \tag{3.4}$$
$$\leq \alpha^t \sum_{i=1}^N \sum_{s \in \mathcal{G}^t} \|v_i(\mathbf{x}^s)\|_i^* \leq \alpha^t \sum_{i=1}^N |\mathcal{G}^t| V_{\max}^i \leq \alpha^t D V_{\max} \tag{3.5}$$

By definition, we can expand b_i^t as follows:

$$b_i^t = \sum_{s\in\mathcal{G}^t}\{v_i(\mathbf{x}^s) - v_i(\mathbf{x}^t)\} \leq \sum_{s\in\mathcal{G}^t} L\|\mathbf{x}^s - \mathbf{x}^t\| = \sum_{s\in\mathcal{G}^t} L\|C(\mathbf{y}^s) - C(\mathbf{y}^t)\| \leq \sum_{s\in\mathcal{G}^t}\frac{L}{K}\|\mathbf{y}^s - \mathbf{y}^t\|^* \tag{3.6}$$

$$\leq \frac{L}{K}\sum_{s\in\mathcal{G}^t}\left\{\|\mathbf{y}^s - \mathbf{y}^{s+1}\|^* + \|\mathbf{y}^{s+1} - \mathbf{y}^{s+2}\|^* + \cdots + \|\mathbf{y}^{t-1} - \mathbf{y}^t\|^*\right\} \tag{3.7}$$

$$\leq \frac{L}{K}\sum_{s\in\mathcal{G}^t}\left\{\alpha^s DV_{\max} + \alpha^{s+1}DV_{\max} + \cdots + \alpha^t DV_{\max}\right\} \tag{3.8}$$

$$= \frac{LDV_{\max}}{K}\sum_{s\in\mathcal{G}^t}\sum_{k=s}^{t}\alpha^k \leq \frac{LDV_{\max}}{K}|\mathcal{G}^t|\sum_{s=\min\mathcal{G}^t}^{t}\alpha^s \leq \frac{LD^2V_{\max}}{K}\sum_{s=\min\mathcal{G}^t}^{t}\alpha^s \tag{3.9}$$

$$\leq \frac{LD^2V_{\max}}{K}\sum_{s=t-D+1}^{t}\alpha^s \leq \frac{LD^3V_{\max}}{K}\alpha^{t-D+1} \to 0, \text{as } t\to\infty, \tag{3.10}$$

where the first inequality in Equation 3.6 follows from the fact that $v(\cdot)$ is L-Lipschitz continuous, the second inequality in Equation 3.6 follows from the fact that $C(\cdot)$ is $\frac{1}{K}$-Lipschitz continuous, Equation 3.8 follows from Equations 3.4 and 3.5 and the first inequality in Equation 3.9 follows from that α^t's are non-negative and the second inequality in Equation 3.10 follows from α^t is non-increasing. Lastly, the convergence to 0 in Equation 3.10 follows from the fact that α^t is square-summable.

2. Fix an arbitrary $\epsilon > 0$. Assume for contradiction purposes that $\mathbf{x}^t$ only visits $B(\mathbf{x}^*, \epsilon)$ a finite number of times and hence let $t^0 - 1$ be the last time $\mathbf{x}^t$ is in $B(\mathbf{x}^*, \epsilon)$: $\forall t \geq t^0, \mathbf{x}^t \in \mathcal{X} - B(\mathbf{x}^*, \epsilon)$. Since $\mathcal{X} - B(\mathbf{x}^*, \epsilon)$ is a compact set and $v_i(\mathbf{x})$ is continuous in $\mathbf{x}$ and since by assumption $\sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - x_i^*) < 0, \forall \mathbf{x}\in\mathcal{X}, \mathbf{x}\neq\mathbf{x}^*$, it follows that there exists a $c_{\max}(\epsilon) < 0$ such that

$$\sum_{i=1}^N \lambda_i v_i(\mathbf{x})(x_i - x_i^*) \leq c_{\max}(\epsilon), \forall \mathbf{x}\in\mathcal{X} - B(\mathbf{x}^*, \epsilon). \tag{3.11}$$

Per Claim 1, $\lim_{t\to\infty}\mathbf{b}^t = \mathbf{0}$, therefore, $\|\mathbf{b}^t\|^*$ is bounded and we denote $B_{\max} \triangleq \max_t \|\mathbf{b}^t\|^*$. Next denote $R = \max_{\mathbf{x}\in\mathcal{X}}\|\mathbf{x}\|$, $\lambda_{\max} \triangleq \max_i \lambda_i$ and $\beta^t \triangleq \max_i \frac{(\alpha^t)^2\lambda_i}{2K_i}$ and note that $\sum_{t=1}^\infty \beta^t < \infty$. Using Lemma 3.1, we have $\forall t \geq t^0$:

$$F^\lambda(\mathbf{x}^*, \mathbf{y}^{t+1}) = F^\lambda(\mathbf{x}^*, \mathbf{y}^t + \alpha^t\{|\mathcal{G}^t|v(\mathbf{x}^t) + \mathbf{b}^t\}) \leq \tag{3.12}$$

$$F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \sum_{i=1}^N \lambda_i(\alpha^t\{|\mathcal{G}^t|v_i(\mathbf{x}^t) + b_i^t\})(C_i(y_i^t) - x_i^*) + \beta^t(\||\mathcal{G}^t|v(\mathbf{x}^t) + \mathbf{b}^t\|^*)^2 = \tag{3.13}$$

$$F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \alpha^t\left\{|\mathcal{G}^t|\sum_{i=1}^N \lambda_i v_i(\mathbf{x}^t)(x_i^t - x_i^*) + \sum_{i=1}^N \lambda_i b_i^t(x_i^t - x_i^*)\right\} + \beta^t(\||\mathcal{G}^t|v(\mathbf{x}^t) + \mathbf{b}^t\|^*)^2 \tag{3.14}$$

$$\leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \alpha^t\left\{|\mathcal{G}^t|c_{\max}(\epsilon) + \lambda_{\max}\|\mathbf{b}^t\|^*\|\mathbf{x}^t - \mathbf{x}^*\|\right\} + \beta^t\left\{2(\||\mathcal{G}^t|v(\mathbf{x}^t)\|^*)^2 + 2(\|\mathbf{b}^t\|^*)^2\right\} \tag{3.15}$$

$$\leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \alpha^t\left\{|\mathcal{G}^t|c_{\max}(\epsilon) + \lambda_{\max}R\|\mathbf{b}^t\|^*\right\} + 2\beta^t(D^2V_{\max}^2 + B_{\max}^2) \tag{3.16}$$

$$\leq F^\lambda(\mathbf{x}^*, \mathbf{y}^{t^0}) + (\sum_{k=t^0}^t \alpha^k)\left\{\frac{\sum_{k=t^0}^t \alpha^k|\mathcal{G}^k|}{\sum_{k=t^0}^t \alpha^k}c_{\max}(\epsilon) + \lambda_{\max}R\frac{\sum_{k=t^0}^t \alpha^k\|\mathbf{b}^k\|^*}{\sum_{k=t^0}^t \alpha^k}\right\} \tag{3.17}$$

$$+ 2(\sum_{k=t^0}^t \beta^k)(D^2V_{\max}^2 + B_{\max}^2), \tag{3.18}$$

where Equation (3.15) follows from Equation (3.11) and Equation (3.17) follows from telescoping.

Next, we show that:

$$1 \le \lim_{t\to\infty} \frac{\sum_{k=t^0}^{t} \alpha^k |\mathcal{G}^k|}{\sum_{k=t^0}^{t} \alpha^k} \le D. \tag{3.19}$$

Partition the rounds into intervals $\{0, 1, \ldots, D-1\}, \{D, D+1, \ldots, 2D-1\}, \ldots$. Since each gradient is received exactly once with at most delay D, the gradients corresponding to the first interval will have been completely received by time $2D-1$ (i.e. by the end of the second interval). Similarly, the gradients corresponding to the l-th interval will have been all received by time $(l+1)D-1$. Consequently, since α^t is non-increasing, it follows that:

$$\sum_{k=0}^{\infty} \alpha^k |\mathcal{G}^k| \ge \sum_{l=2}^{\infty} D\alpha^{lD-1} \ge \sum_{k=2D-1}^{\infty} \alpha^k = \infty.$$

Consequently,

$$\lim_{t\to\infty} \frac{\sum_{k=t^0}^{t} \alpha^k |\mathcal{G}^k|}{\sum_{k=t^0}^{t} \alpha^k} = \lim_{t\to\infty} \frac{\sum_{k=0}^{t} \alpha^k |\mathcal{G}^k|}{\sum_{k=0}^{t} \alpha^k} = \lim_{t\to\infty} \frac{\sum_{k=0}^{t} \alpha^k |\mathcal{G}^k|}{\sum_{k=2D-1}^{t} \alpha^k} \ge 1.$$

Finally, $\frac{\sum_{k=t^0}^{t} \alpha^k |\mathcal{G}^t|}{\sum_{k=t^0}^{t} \alpha^k} \le D$ follows easily by noting that $|\mathcal{G}^t| \le D$.

Next, note that since $\lim_{t\to\infty} \mathbf{b}^t = \mathbf{0}$ and $\sum_{t=0}^{\infty} \alpha^t = \infty$, it follows that:

$$\frac{\sum_{k=t^0}^{t} \alpha^k \|\mathbf{b}^k\|^*}{\sum_{k=t^0}^{t} \alpha^k} \to 0, \text{ as } t \to \infty. \tag{3.20}$$

Combining Equation 3.19 and Equation 3.20, we obtain:

$$(\sum_{k=t^0}^{t} \alpha^k) \left\{ \frac{\sum_{k=t^0}^{t} \alpha^k |\mathcal{G}^k|}{\sum_{k=t^0}^{t} \alpha^k} c_{\max}(\epsilon) + \lambda_{\max} R \frac{\sum_{k=t^0}^{t} \alpha^k \|\mathbf{b}^k\|^*}{\sum_{k=t^0}^{t} \alpha^k} \right\} \to -\infty, \text{ as } t \to \infty.$$

Since $\sum_{k=t^0}^{\infty} \beta^k < \infty$, Equation (3.17) implies that $F^\lambda(\mathbf{x}^*, \mathbf{y}^t) \to -\infty$, which contradicts the first statement in Lemma 3.1. The claim therefore follows.

3. Assume for contradiction purposes no $B(\mathbf{x}^*, \epsilon)$ is contained in $\tilde{B}(\mathbf{x}^*, \delta)$, which means that for any $\delta > 0, \exists \mathbf{y}^l$, such that $\|Q(\mathbf{y}^l) - \mathbf{x}^*\| = \delta$ but $F^\lambda(\mathbf{x}^*, \mathbf{y}^l) \ge \epsilon$. This produces a sequence $\{\mathbf{y}^l\}_{l=0}^{\infty}$ such that $C(\mathbf{y}^l) \to \mathbf{x}^*$ but $F^\lambda(\mathbf{x}^*, \mathbf{y}^l) \ge \epsilon, \forall l$. This contradicts with the fact that the choice map $C(\cdot)$ is λ-Fenchel coupling conforming, because by definition it holds that if $C(\mathbf{y}^t) \to x$, then $F^\lambda(x, \mathbf{y}^t) \to 0$. Consequently, the claim follows.

4. Fix a given $\delta > 0$. Since β^t is summable and α^t is not summable but square summable, it follows that $\beta^t \to 0, \alpha^t \to 0, \frac{\alpha^t}{\beta^t} \to \infty$ as $t \to \infty$. There, denote

   (a) $T^1(\delta) = \arg\min_t \{t \mid \beta^s < \frac{\delta}{8(D^2 V_{\max}^2 + B_{\max}^2)}, \forall s \ge t\}$.

   (b) $T^2(\delta) = \arg\min_t \{t \mid c_{\max}(\epsilon(\frac{\delta}{2})) < -2\lambda_{\max} R \|\mathbf{b}^s\|^*, \forall s \ge t\}$.

   (c) $T^3(\delta) = \arg\min_t \{t \mid \alpha^s < \frac{\delta}{4\lambda_{\max} R B_{\max}}, \forall s \ge t\}$.

   (d) $T^4(\delta) = \arg\min_t \{t \mid \frac{\alpha^s}{\beta^s} > \frac{4(D^2 V_{\max}^2 + B_{\max}^2)}{-c_{\max}(\epsilon(\frac{\delta}{2}))}, \forall s \ge t\}$.

   We have $T^1(\delta) < \infty, T^2(\delta) < \infty$ (since $\lim_{t\to\infty} \|\mathbf{b}^t\|^* = 0$ and note that $c_{\max}(\epsilon(\frac{\delta}{2})) < 0$ by definition), $T^3(\delta) < \infty, T^4(\delta) < \infty$. Take

   $$T(\delta) = \max(T^1(\delta), T^2(\delta), T^3(\delta), T^4(\delta)\}.$$

   Now take any $t \ge T(\delta)$. We show that if $\mathbf{x}^t \in \tilde{B}(\mathbf{x}^*, \delta)$, then $\mathbf{x}^{t+1} \in \tilde{B}(\mathbf{x}^*, \delta)$. To see that this statement holds, let $\mathbf{x}^t \in \tilde{B}(\mathbf{x}^*, \delta)$, which implies that $F^\lambda(\mathbf{x}^*, \mathbf{y}^t) < \delta$. Note that it suffices to consider $\mathcal{G}^t \neq \emptyset$, for otherwise $\mathbf{x}^{t+1} = \mathbf{x}^t$.

   Now there are two possibilities:

(a) Possibility 1: $\mathbf{x}^t \in B(\mathbf{x}^*, \epsilon(\frac{\delta}{2}))$.

(b) Possibility 2: $\mathbf{x}^t \in \tilde{B}(\mathbf{x}^*, \delta) - B(\mathbf{x}^*, \epsilon(\frac{\delta}{2}))$.

Under Possibility 1, it follows

$$F^\lambda(\mathbf{x}^*, \mathbf{y}^{t+1}) \leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \alpha^t \sum_{i=1}^{N} \lambda_i \{|\mathcal{G}^t| v_i(\mathbf{x}^t) + b_i^t\}(x_i^t - x_i^*) + \beta^t (\||\mathcal{G}^t| v(\mathbf{x}_t) + \mathbf{b}^t\|^*)^2 \tag{3.21}$$

$$\leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \alpha^t \sum_{i=1}^{N} \lambda_i b_i^t (x_i^t - x_i^*) + \beta^t \left\{2(\||\mathcal{G}^t| v(\mathbf{x}_t)\|^*)^2 + 2(\|\mathbf{b}^t\|^*)^2\right\} \tag{3.22}$$

$$\leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \alpha^t \lambda_{\max} R B_{\max} + 2\beta^t (D^2 V_{\max}^2 + B_{\max}^2) \tag{3.23}$$

$$< F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \frac{\delta}{4\lambda_{\max} R B_{\max}} \lambda_{\max} R B_{\max} + \frac{2\delta}{8(D^2 V_{\max}^2 + B_{\max}^2)} (D^2 V_{\max^2} + B_{\max}^2) \tag{3.24}$$

$$\leq \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta, \tag{3.25}$$

where the second inequality follows from λ-variational stability and the last inequality follows from the fact that $\mathbf{x}^t \in B(\mathbf{x}^*, \epsilon(\frac{\delta}{2})) \subset \tilde{B}(\mathbf{x}^*, \frac{\delta}{2})$ per Claim 2.

Under Possibility 2, it follows from Equation 3.16 that

$$F^\lambda(\mathbf{x}^*, \mathbf{y}^{t+1}) \leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + \alpha^t \left\{ |\mathcal{G}^t| c_{\max}(\epsilon(\frac{\delta}{2})) + \lambda_{\max} R \|\mathbf{b}^t\|^* \right\} + 2\beta^t (D^2 V_{\max}^2 + B_{\max}^2) \tag{3.26}$$

$$\leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + 2\beta^t (D^2 V_{\max}^2 + B_{\max}^2) \left\{ \frac{\alpha^t}{\beta^t} \frac{c_{\max}(\epsilon(\frac{\delta}{2})) + \lambda_{\max} R \|\mathbf{b}^t\|^*}{2(D^2 V_{\max}^2 + B_{\max}^2)} + 1 \right\} \tag{3.27}$$

$$\leq F^\lambda(\mathbf{x}^*, \mathbf{y}^t) + 2\beta^t (V_{\max}^2 + B_{\max}^2) \left\{ \frac{\alpha^t}{\beta^t} \frac{c_{\max}(\epsilon(\frac{\delta}{2}))}{4(V_{\max}^2 + B_{\max}^2)} + 1 \right\} \tag{3.28}$$

$$< F^\lambda(\mathbf{x}^*, \mathbf{y}^t) < \epsilon, \tag{3.29}$$

where the second inequality follows from $|\mathcal{G}^t| \geq 1$ since it is not empty by assumption and $c_{\max} < 0$, the third inequality follows from $\lambda_{\max} R \|\mathbf{b}^t\|^* < -\frac{1}{2} c_{\max}(\epsilon(\frac{\delta}{2}))$ since $t \geq T^2(\delta)$ and the second-to-last inequality follows from $\frac{\alpha^t}{\beta^t} \frac{c_{\max}(\epsilon(\frac{\delta}{2}))}{4(V_{\max}^2 + B_{\max}^2)} + 1 < 0$ since $t \geq T^4(\delta)$.

■

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