Supplementary Material: Multi-objective Maximization of Monotone Submodular Functions with Cardinality Constraint

1 Some More Notation and Preliminaries

Let $\beta(\eta) = 1 - \frac{1}{e^\eta} \in [0, 1 - 1/e]$ for $\eta \in [0,1]$. Note that $\beta(1) = (1 - 1/e)$. Further, for $k' \leq k$,

$$\beta(k'/k) = (1 - e^{1-k'/k}/e) \geq (1 - 1/e)k'/k. \tag{1}$$

This function appears naturally in our analysis and will be useful for expressing approximation guarantees. Next, the lemma below formalizes Stage 2 of the algorithm in [CVZ10].

Lemma 8. *([CVZ10] Lemma 7.3) Given submodular functions f_i and values V_i, cardinality k, the continuous greedy algorithm finds a point $\mathbf{x} \in [0,1]^n$ such that $F_i(\mathbf{x}(k)) \geq (1 - 1/e - \epsilon')V_i\, \forall i$ with $\epsilon' = 1/\Omega(k)$, or outputs a certificate of infeasibility.*

2 Missing Proofs from Section 3.1

Corollary 9. *Given a point $\mathbf{x} \in [0,1]^n$ with $|\mathbf{x}| = k$ and a multilinear extension F of a monotone submodular function, for every $k_1 \leq k$,*

$$F\Big(\frac{k_1}{k}\mathbf{x}\Big) \geq \frac{k_1}{k} F(\mathbf{x}).$$

Proof. Note that the statement is true for concave F. The proof now follows directly from the concavity of multilinear extensions in positive directions (Section 2.1 of [CCPV11]). □

Lemma 10. $F_i(\mathbf{x}(k_1)|\mathbf{x}_{S_1}) \geq (\beta(1) - \epsilon')\frac{k_1}{k}(V_i - f_i(S_1))$ *for every i.*

Proof. Recall that S_k denotes a feasible solution with cardinality k, and let $\mathbf{x}_{S_k}$ denote its characteristic vector. Clearly, $|\mathbf{x}_{S_k \setminus S_1}| \leq k$ and $F_i(\mathbf{x}_{S_k \setminus S_1}|\mathbf{x}_{S_1}) = f_i(S_k|S_1) \geq (V_i - f_i(S_1))$ for very i. And now from Corollary 9, we have that there exists a point $\mathbf{x}'$ with $|\mathbf{x}'| = k_1$ such that $F_i(\mathbf{x}'|\mathbf{x}_{S_1}) \geq \frac{k_1}{k} F_i(\mathbf{x}_{S_k \setminus S_1}|\mathbf{x}_{S_1})$ for every i. Finally, using Lemma 8 we have $F_i(\mathbf{x}(k_1)|\mathbf{x}_{S_1}) \geq (\beta(1) - \epsilon')F_i(\mathbf{x}'|S_1)$, which gives the desired bound. □

3 Missing Proofs from Section 3.2

Lemma 3. $g^t(X^t) \geq \frac{k_1}{k}\alpha \sum_i \lambda_i^t, \forall t.$

Proof. Consider the optimal set S_k and note that $\sum_i \lambda_i^t \tilde{f}_i(S_k) \geq \sum_i \lambda_i^t, \forall t$. Now the function $g^t(.) = \sum_i \lambda_i^t \tilde{f}_i(.)$, being a convex combination of monotone submodular functions, is also monotone submodular. We would like to show that there exists a set S' of size k_1 such that $g^t(S') \geq \frac{k_1}{k}\sum_i \lambda_i^t$. Then the claim follows from the fact that $\mathcal{A}$ is an α approximation for monotone submodular maximization with cardinality constraint.

To see the existence of such a set S', greedily index the elements of S_k using $g^t(.)$. Suppose that the resulting order is $\{s_1, \ldots, s_k\}$, where s_i is such that $g^t(s_i|\{s_1, \ldots, s_{i-1}\}) \geq g^t(s_j|\{s_1, \ldots, s_{i-1}\})$ for every $j > i$. Then the truncated set $\{s_1, \ldots, s_{k-|S_1|}\}$ has the desired property, and we are done. □

Lemma 4.

$$\frac{\sum_t \tilde{f}_i(X^t)}{T} \geq \frac{k_1}{k}(1 - 1/e) - \delta\,, \forall i.$$

Proof. Suppose we have,

$$\frac{\sum_t \tilde{f}_i(X^t) - \alpha}{T} + \delta \geq \frac{1}{T}\sum_t \sum_i \frac{\lambda_i^t}{\sum_i \lambda_i^t}(\tilde{f}_i(X^t) - \alpha), \forall i. \tag{2}$$

Then assuming $\alpha = (1-1/e)$, the RHS above simplifies to,

$$\frac{1}{T}\sum_t \frac{g(X^t)}{\sum_i \lambda_i^t} - (1-1/e) \qquad \geq (1-1/e)(\frac{k_1}{k}-1) \quad \text{(using Lemma 3)}$$

And we have for every i,

$$\begin{aligned} \frac{\sum_t \tilde{f}_i(X^t) - (1-1/e)}{T} + \delta \qquad &\geq (1-1/e)(\frac{k_1}{k}-1) \\ \frac{\sum_t \tilde{f}_i(X^t)}{T} \qquad &\geq \frac{k_1}{k}(1-1/e) - \delta. \end{aligned}$$

Now, the proof for (2) closely resembles the analysis in Theorem 3.3 and 2.1 in **(author?)** AHK12. We will use the potential function $\Phi^t = \sum_i \lambda_i^t$. Let $p_i^t = \lambda_i^t/\Phi^t$ and $M^t = \sum_i p_i^t m_i^t$. Then we have,

$$\begin{aligned} \Phi^{t+1} \qquad &= \sum_i \lambda_i^t(1-\delta m_i^t) \\ &= \Phi^t - \delta\Phi^t \sum_i p_i^t m_i^t \\ &= \Phi^t(1-\delta M^t) \leq \Phi^t e^{-\delta M^t} \end{aligned}$$

After T rounds, $\Phi^T \leq \Phi^1 e^{-\delta\sum_t M^t}$. Further, for every i,

$$\begin{gathered} \Phi^T \geq w_i^T = \tfrac{1}{m}\textstyle\prod_t(1-\delta m_i^t) \\ \ln(\Phi^1 e^{-\delta\sum_t M^t}) \geq \textstyle\sum_t \ln(1-\delta m_i^t) - \ln m \\ \delta \textstyle\sum_t M^t \leq \ln m + \sum_t \ln(1-\delta m_i^t) \end{gathered}$$

Using $\ln(\frac{1}{1-\epsilon}) \leq \epsilon + \epsilon^2$ and $\ln(1+\epsilon) \geq \epsilon - \epsilon^2$ for $\epsilon \leq 0.5$, and with $T = \frac{2\ln m}{\delta^2}$ and $\delta < (1-1/e)$ (for a positive approximation guarantee), we have,

$$\frac{\sum_t M^t}{T} \leq \delta + \frac{\sum_t m_i^t}{T}, \forall i.$$

$\square$

Lemma 5. *Given monotone submodular function f, its multilinear extension F, sets X^t for $t \in \{1,\ldots,T\}$, and a point $\mathbf{x} = \sum_t X^t/T$, we have,*

$$F(\mathbf{x}) \geq (1-1/e)\frac{1}{T}\sum_{t=1}^{T} f(X^t).$$

Proof. Consider the concave closure of a submodular function f,

$$f^+(\mathbf{x}) = \max_\alpha\{\sum_X \alpha_X f(X) | \sum_X \alpha_X X = \mathbf{x}, \sum_X \alpha_X \leq 1, \alpha_X \geq 0\, \forall X \subseteq N\}.$$

Clearly, $f_i^+(\mathbf{x}) \geq \frac{\sum_t f_i(X^t)}{T}$. So it suffices to show $F_i(\mathbf{x}) \geq (1-1/e)f_i^+(\mathbf{x})$, which in fact, follows from Lemmas 4 and 5 in [CCPV07].

Alternatively, we now give a novel and direct proof for the statement. We abuse notation and use $\mathbf{x}_{X^t}$ and X^t interchangeably. Let $\mathbf{x} = \sum_{t=1}^T X^t/T$ and w.l.o.g., assume that sets X^t are indexed such that $f(X^j) \geq f(X^{j+1})$ for every $j \geq 1$. Further, let $f(X^t)/T = a^t$ and $\sum_t a^t = A$.

Recall that $F(\mathbf{x})$ can be viewed as the expected function value of the set obtained by independently sampling element j with probability x_j. Instead, consider the alternative random process where starting with $t=1$, one samples each element in set X^t independently with probability $1/T$. The random process runs in T steps and the probability of an element j being chosen at the end of the process is exactly $p_j = 1-(1-1/T)^{Tx_j}$, independent of all other elements. Let $\mathbf{p} = (p_1,\ldots,p_n)$, it follows that the expected value of the set sampled using this process is given by $F(\mathbf{p})$. Observe that for every j, $p_j \leq x_j$ and therefore, $F(\mathbf{p}) \leq F(\mathbf{x})$. Now in step t, suppose the newly sampled subset of X^t adds marginal value Δ^t. From submodularity we have, $\mathbb{E}[\Delta^1] \geq \frac{f(X^1)}{T} = a^1$ and in general, $\mathbb{E}[\Delta^t] \geq \frac{f(X^t) - \mathbb{E}[\sum_{j=1}^{t-1} \Delta_j]}{T} \geq a^t - \frac{1}{T}\sum_{j=1}^{t-1} \mathbb{E}[\Delta^j]$.

To see that $\sum_t \mathbb{E}[\Delta^t] \geq (1-1/e)A$, consider a LP where the objective is to minimize $\sum_t \gamma^t$ subject to $b^1 \geq b^2 \cdots \geq b^T \geq 0$; $\sum b^t = A$ and $\gamma^t \geq b^t - \frac{1}{T}\sum_{j=1}^{t-1} \gamma^j$ with $\gamma^0 = 0$. Here A is a parameter and everything else is a variable. Observe that the extreme points are characterized by j such that, $\sum b^t = A$ and $b^t = b^1$ for all $t \leq j$ and $b^{j+1} = 0$. For all such points, it is not difficult to see that the objective is at least $(1-1/e)A$. Therefore, we have $F(\mathbf{p}) \geq (1-1/e)A = (1-1/e)\sum_t f(X^t)/T$, as desired.

□

4 Missing Proofs from Section 3.3

Lemma 7. *Given that there exists a set S_k such that $f_i(S_k) \geq V_i, \forall i$ and $\epsilon < \frac{1}{8\ln m}$. For every $k' \in [m/\epsilon^3, k]$, there exists $S_{k'} \subseteq S_k$ of size k', such that,*

$$f_i(S_{k'}) \geq (1-\epsilon)\Big(\frac{k' - m/\epsilon^3}{k - m/\epsilon^3}\Big)V_i, \forall i.$$

Proof. We restrict our ground set of elements to S_k and let S_1 be a subset of size at most m/ϵ^3 such that $f_i(e|S_1) < \epsilon^3 V_i, \forall e \in S_k \backslash S_1$ and $\forall i$ (recall, we discussed the existence of such a set in Section 2.1, Stage 1). The rest of the proof is similar to the proof of Lemma 10. Consider the point $\mathbf{x} = \frac{k'-|S_1|}{k-|S_1|}\mathbf{x}_{S_k \backslash S_1}$. Clearly, $|\mathbf{x}| = k' - |S_1|$, and from Corollary 9, we have $F_i(\mathbf{x}|\mathbf{x}_{S_1}) \geq \frac{k'-|S_1|}{k-|S_1|}F_i(\mathbf{x}_{S_k\backslash S_1}|\mathbf{x}_{S_1}) = \frac{k'-|S_1|}{k-|S_1|}f_i(S_k \backslash S_1 | S_1) \geq \frac{k'-|S_1|}{k-|S_1|}(V_i - f_i(S_1)), \forall i$. Finally, using swap rounding Lemma 1, there exists a set S_2 of size $k' - |S_1|$, such that $f_i(S_1 \cup S_2) \geq (1-\epsilon)\frac{k'-|S_1|}{k-|S_1|}V_i, \forall i$. □

Theorem 8. *For $k' = \frac{m}{\epsilon^4}$, choosing k'-tuples greedily w.r.t. $h(.) = \min_i f_i(.)$ yields approximation guarantee $(1-1/e)(1-2\epsilon)$ for $k \to \infty$, while making n^{m/ϵ^4} queries.*

Proof. The analysis generalizes that of the standard greedy algorithm ([NW78, NWF78]). Let S_j denote the set at the end of iteration j. $S_0 = \emptyset$ and let the final set be $S_{\lfloor k/k' \rfloor}$. Then from Theorem 7, we have that at step $j+1$, there is some set $X \in S_k \backslash S_j$ of size k' such that

$$f_i(X|S_j) \geq (1-\epsilon)\frac{k' - m/\epsilon^3}{k - m/\epsilon^3}\big(V_i - f_i(S_j)\big), \forall i.$$

To simplify presentation let $\eta = (1-\epsilon)\frac{k'-m/\epsilon^3}{k-m/\epsilon^3}$ and note that $\eta \leq 1$. Further, $1/\eta \to \infty$ as $k \to \infty$ for fixed m and $k' = o(k)$. Now, we have for every i, $f_i(S_{j+1}) - (1-\eta)f_i(S_j) \geq \eta V_i$. Call this inequality $j+1$. Observe that inequality $\lfloor k/k' \rfloor$ states $f_i(S_{\lfloor k/k' \rfloor}) - (1-\eta)f_i(S_{\lfloor k/k' \rfloor - 1}) \geq \eta V_i, \forall i$. Therefore, multiplying inequality $\lfloor k/k' \rfloor - j$ by $(1-\eta)^j$ and telescoping over j we get for every i,

$$\begin{aligned} f_i(S_{\lfloor k/k' \rfloor}) \quad &\geq \sum_{j=0}^{\lfloor k/k' \rfloor - 1} (1-\eta)^j \eta V_i \\ &\geq \big(1 - (1-\eta)^{\lfloor k/k' \rfloor}\big)V_i \\ &\geq \big(1 - (1-\eta)^{\frac{1}{\eta}\eta\lfloor k/k' \rfloor}\big)V_i \\ &\geq \beta(\eta\lfloor k/k' \rfloor)V_i \quad \geq (1-1/e)(\eta\lfloor k/k' \rfloor)V_i. \end{aligned}$$

Where we used (1) for the last inequality. Let $\epsilon = \sqrt[4]{\frac{m}{k'}}$, then we have,

$$\eta\lfloor k/k' \rfloor \geq (1-\epsilon)\frac{1 - m/k'\epsilon^3}{1 - m/k\epsilon^3}\Big(1 - \frac{k'}{k}\Big) \geq \frac{\Big(1 - \sqrt[4]{\frac{m}{k'}}\Big)^2}{1 - \frac{1}{k}\sqrt[4]{\frac{m}{(k')^3}}}\Big(1 - \frac{k'}{k}\Big)$$

As $k \to \infty$ we get the asymptotic guarantee $(1-1/e)\Big(1 - \sqrt[4]{\frac{m}{k'}}\Big)^2 = (1-1/e)(1-\epsilon)^2$. □

References

[AHK12] S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory of Computing*, 8:121–164, 2012.

[CCPV07] G. Calinescu, C. Chekuri, Martin Pál, and J. Vondrák. Maximizing a submodular set function subject to a matroid constraint. In *IPCO*, volume 7, pages 182–196, 2007.

[CCPV11] G. Calinescu, C. Chekuri, M. Pál, and J. Vondrák. Maximizing a monotone submodular function subject to a matroid constraint. *SIAM Journal on Computing*, 40(6):1740–1766, 2011.

[CVZ10] C. Chekuri, J. Vondrák, and R. Zenklusen. Dependent randomized rounding via exchange properties of combinatorial structures. In *FOCS 10*, pages 575–584. IEEE, 2010.

[NW78] G.L. Nemhauser and L.A. Wolsey. Best algorithms for approximating the maximum of a submodular set function. *Mathematics of operations research*, 3(3):177–188, 1978.

[NWF78] G.L. Nemhauser, L.A. Wolsey, and M.L. Fisher. An analysis of approximations for maximizing submodular set functions—i. *Mathematical Programming*, 14(1):265–294, 1978.