
Multi-Objective Non-parametric Sequential Prediction

1 Proofs of helping lemmas

Lemma 2 (Continuity and Minimax). *Let $\mathcal{Y}, \Lambda, \mathcal{X}$ be compact real spaces. $l : \mathcal{Y} \times \Lambda \times \mathcal{X} \rightarrow \mathbb{R}$ be a continuous function. Denote by $\mathbb{P}(\mathcal{X})$ the space of all probability measures on \mathcal{X} (equipped with the topology of weak-convergence). Then the following function $L^* : \mathbb{P}(\mathcal{X}) \rightarrow \mathbb{R}$ is continuous*

$$L^*(\mathbb{Q}) = \inf_{y \in \mathcal{Y}} \sup_{\lambda \in \Lambda} \mathbb{E}_{\mathbb{Q}} [l(y, \lambda, x)]. \quad (1)$$

Moreover, for any $\mathbb{Q} \in \mathbb{P}(\mathcal{X})$,

$$\inf_{y \in \mathcal{Y}} \sup_{\lambda \in \Lambda} \mathbb{E}_{\mathbb{Q}} [l(y, \lambda, x)] = \sup_{\lambda \in \Lambda} \inf_{y \in \mathcal{Y}} \mathbb{E}_{\mathbb{Q}} [l(y, \lambda, x)].$$

Proof. $\mathcal{Y}, \Lambda, \mathcal{X}$ are compact, implying that the function $l(y, \lambda, x)$ is bounded. Therefore, the function $L : \mathcal{Y} \times \Lambda \times \mathbb{P}(\mathcal{X}) \rightarrow \mathbb{R}$, defined as

$$L(y, \lambda, \mathbb{Q}) = \mathbb{E}_{\mathbb{Q}} [l(y, \lambda, x)], \quad (2)$$

is continuous. By applying Proposition 7.32 from [2], we have that $\sup_{\lambda \in \Lambda} \mathbb{E}_{\mathbb{Q}} [l(y, \lambda, X)]$ is continuous in $\mathbb{Q} \times \mathcal{Y}$. Again applying the same proposition, we get the desired result. The last part of the lemma follows directly from Fan's minimax theorem [3]. \square

Lemma 3 (Continuity of the optimal selection). *Let $\mathcal{Y}, \Lambda, \mathcal{X}$ be compact real spaces, and let L be as defined in Equation (2). Then, there exist two measurable selection functions h^y, h^λ such that*

$$h^y(\mathbb{Q}) \in \arg \min_{y \in \mathcal{Y}} \left(\max_{\lambda \in \Lambda} L(y, \lambda, \mathbb{Q}) \right),$$

$$h^\lambda(\mathbb{Q}) \in \arg \max_{\lambda \in \Lambda} \left(\min_{y \in \mathcal{Y}} L(y, \lambda, \mathbb{Q}) \right)$$

for any $\mathbb{Q} \in \mathbb{P}(\mathcal{X})$. Moreover, let L^* be as defined in Equation (1). Then, the set

$$Gr(L^*) \triangleq \{(u^*, v^*, \mathbb{Q}) \mid u^* \in h^y(\mathbb{Q}), v^* \in h^\lambda(\mathbb{Q}), \mathbb{Q} \in \mathbb{P}(\mathcal{X})\},$$

is closed in $\mathcal{Y} \times \Lambda \times \mathbb{P}(\mathcal{X})$.

Proof. The first part of the proof follows immediately from the minimax measurable theorem of [8] due to the compactness of $\mathcal{Y}, \Lambda, \mathcal{X}$ and the properties of the loss function L . The proof of the second part is similar to the one presented in Theorem 3 of [1]. In order to show that $Gr(L^*)$ is closed, it is enough to show that if (i) $\mathbb{Q}_n \rightarrow \mathbb{Q}_\infty$ in $\mathbb{P}(\mathcal{X})$; (ii) $u_n \rightarrow u_\infty$ in \mathcal{Y} ; (iii) $v_n \rightarrow v_\infty$ in Λ and (iv) $u_n \in h^y(\mathbb{Q}_n), v_n \in h^\lambda(\mathbb{Q}_n)$ for all n , then,

$$u_\infty \in h^y(\mathbb{Q}_\infty), v_\infty \in h^\lambda(\mathbb{Q}_\infty).$$

The function $L(y, \lambda, \mathbb{Q})$, as defined in Equation (2), is continuous. Therefore,

$$\lim_{n \rightarrow \infty} L(u_n, v_n, \mathbb{Q}_n) = L(u_\infty, v_\infty, \mathbb{Q}_\infty).$$

It remains to show that $u_\infty \in h^y(\mathbb{Q}_\infty)$ and $v_\infty \in h^\lambda(\mathbb{Q}_\infty)$. From the optimality of u_n and v_n , we obtain

$$L(u_\infty, v_\infty, \mathbb{Q}_\infty) = \lim_{n \rightarrow \infty} L(u_n, v_n, \mathbb{Q}_n) = \lim_{n \rightarrow \infty} L^*(\mathbb{Q}_n). \quad (3)$$

Finally, from the continuity of L^* (Lemma 2), we get

$$(3) = L^*(\lim_{n \rightarrow \infty} \mathbb{Q}_n) = L^*(\mathbb{Q}_\infty),$$

which gives the desired result. \square

Corollary 1. *Under the conditions of Lemma 3. Define $L_n(y, \lambda, \mathbb{Q}) = L(y, \lambda, \mathbb{Q}) + \frac{\|y\|^2 - \|\lambda\|^2}{n}$ and denote $h_{L_n}^y(\mathbb{Q}_n), h_{L_n}^\lambda(\mathbb{Q}_n)$ to be the measurable selection functions of L_n . If $\mathbb{Q}_n \rightarrow \mathbb{Q}_\infty$ weakly in $\mathbb{P}(\mathcal{X})$ and $u_n \in h_{L_n}^y(\mathbb{Q}_n), v_n \in h_{L_n}^\lambda(\mathbb{Q}_n)$, then*

$$L_n(u_n, v_n, \mathbb{Q}_n) \rightarrow L(u_\infty, v_\infty, \mathbb{Q}_\infty)$$

almost surely for $u_\infty \in h^y(\mathbb{Q}_\infty)$ and $v_\infty \in h^\lambda(\mathbb{Q}_\infty)$.

Proof. Denote $\hat{u}_n \in h^y(\mathbb{Q}_\infty)$ and $\hat{v}_n \in h^\lambda(\mathbb{Q}_\infty)$

$$\begin{aligned} & |L_n(u_n, v_n, \mathbb{Q}_n) - L(u_\infty, v_\infty, \mathbb{Q}_\infty)| \\ & \leq |L_n(u_n, v_n, \mathbb{Q}_n) - L(\hat{u}_n, \hat{v}_n, \mathbb{Q}_n)| + |L(\hat{u}_n, \hat{v}_n, \mathbb{Q}_n) - L(u_\infty, v_\infty, \mathbb{Q}_\infty)|. \end{aligned} \quad (4)$$

Note that for every n and for constant $E > 0$,

$$\begin{aligned} & \min_{y \in \mathcal{Y}} \max_{\lambda \in \Lambda} L(y, \lambda, \mathbb{Q}) - \frac{\|\lambda_{\max}\|^2}{n} \leq \min_{y \in \mathcal{Y}} \max_{\lambda \in \Lambda} L_n(y, \lambda, \mathbb{Q}) \\ & = \min_{y \in \mathcal{Y}} \max_{\lambda \in \Lambda} \left(\mathbb{E}_{\mathbb{Q}} [l(y, \lambda, X)] + \frac{\|y\|^2 - \|\lambda\|^2}{n} \right) \\ & \leq \min_{y \in \mathcal{Y}} \max_{\lambda \in \Lambda} L(y, \lambda, \mathbb{Q}) + \frac{E}{n}. \end{aligned}$$

Thus, for some constant C , $|L_n(u_n, v_n, \mathbb{Q}_n) - L(u_\infty, v_\infty, \mathbb{Q}_\infty)| < \frac{C}{n}$ and from Lemma 3, the last summand also converges to 0 as n approaches ∞ , we get the desired result, and clearly, if $h^y(\mathbb{Q}_\infty)$ and $h^\lambda(\mathbb{Q}_\infty)$ are singletons, then, the only accumulation point of $\{(v_n, u_n)\}_{n=1}^\infty$ is (v_∞, u_∞) . \square

2 Proof of Theorem 1

Theorem 1 (Optimality of \mathcal{V}^*). *Let $\{X_i\}_{-\infty}^\infty$ be a γ -feasible process. Then, for any strategy $\mathcal{S} \in \mathcal{S}_\gamma$, the following holds a.s.*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N u(\mathcal{S}(X_1^{i-1}), X_i) \geq \mathcal{V}^*.$$

Proof. For any given strategy $\mathcal{S} \in \mathcal{S}_\gamma$, we will look at the following sequence:

$$\frac{1}{N} \sum_{i=1}^N l(\mathcal{S}(X_1^{i-1}), \tilde{\lambda}_i^*, X_i). \quad (5)$$

where $\tilde{\lambda}_i^* \in h^\lambda(\mathbb{P}_{X_i|X_1^{i-1}})$. Observe that

$$\begin{aligned} (5) & = -\frac{1}{N} \sum_{i=1}^N \left(l(\mathcal{S}(X_1^{i-1}), \tilde{\lambda}_i^*, X_i) - \mathbb{E} \left[l(\mathcal{S}(X_1^{i-1}), \tilde{\lambda}_i^*, X) \mid X_1^{i-1} \right] \right) \\ & \quad + \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[l(\mathcal{S}(X_1^{i-1}), \tilde{\lambda}_i^*, X_i) \mid X_1^{i-1} \right]. \end{aligned}$$

Since $A_i = l(S(X_1^{i-1}), \tilde{\lambda}_i^*, X_i) - \mathbb{E} [l(S(X_1^{i-1}), \tilde{\lambda}_i^*, X_i) | X_1^{i-1}]$ is a martingale difference sequence, the last summand converges to 0 a.s., by the strong law of large numbers (see, e.g., [9]). Therefore,

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \tilde{\lambda}_i^*, X_i) &= \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E} [l(S(X_1^{i-1}), \tilde{\lambda}_i^*, X_i) | X_1^{i-1}] \\ &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \min_{y \in \mathcal{Y}(\cdot)} \mathbb{E} [l(y, \tilde{\lambda}_i^*, X_i) | X_1^{i-1}], \end{aligned} \quad (6)$$

where the minimum is taken w.r.t. all the $\sigma(X_1^{i-1})$ -measurable functions. Because the process is stationary, we get for $\hat{\lambda}_i^* \in h^\lambda(\mathbb{P}_{X_0|X_1^{i-1}})$,

$$(6) = \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \min_{y \in \mathcal{Y}(\cdot)} \mathbb{E} [l(y, \hat{\lambda}_i^*, X_0) | X_1^{i-1}] \quad (7)$$

$$= \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N L^*(\mathbb{P}_{X_0|X_1^{i-1}}). \quad (8)$$

Using Levy's zero-one law, $\mathbb{P}_{X_0|X_1^{i-1}} \rightarrow \mathbb{P}_\infty$ weakly as i approaches ∞ and from Lemma 2 we know that L^* is continuous. Therefore, we can apply Lemma 1 and get that a.s.

$$(8) = \mathbb{E} [L^*(\mathbb{P}_\infty)] = \mathbb{E} [\mathbb{E}_{\mathbb{P}_\infty} [l(y_\infty^*, \lambda_\infty^*, X_0)]] = \mathbb{E} [\mathcal{L}(y_\infty^*, \lambda_\infty^*, X_0)]. \quad (9)$$

Note also, that due to the complementary slackness condition of the optimal solution, i.e., $\lambda_\infty^*(\mathbb{E}_{\mathbb{P}_\infty} [c(y_\infty^*, X_0)] - \gamma) = 0$, we get

$$(9) = \mathbb{E} [\mathbb{E}_{\mathbb{P}_\infty} [u(y_\infty^*, X_0)]] = \mathcal{V}^*.$$

From the uniqueness of λ_∞^* , and using Lemma 3 $\hat{\lambda}_i^* \rightarrow \lambda_\infty^*$ as i approaches ∞ . Moreover, since l is continuous on a compact set, l is also uniformly continuous. Therefore, for any given $\epsilon > 0$, there exists $\delta > 0$, such that if $|\lambda' - \lambda| < \delta$, then

$$|l(y, \lambda', x) - l(y, \lambda, x)| < \epsilon$$

for any $y \in \mathcal{Y}$ and $x \in \mathcal{X}$. Therefore, there exists i_0 such that if $i > i_0$ then $|l(y, \hat{\lambda}_i^*, x) - l(y, \lambda_\infty^*, x)| < \epsilon$ for any $y \in \mathcal{Y}$ and $x \in \mathcal{X}$. Thus,

$$\begin{aligned} &\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \lambda_\infty^*, X_i) - \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \hat{\lambda}_i^*, X_i) \\ &= \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \lambda_\infty^*, X_i) + \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N -l(S(X_1^{i-1}), \hat{\lambda}_i^*, X_i) \\ &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \hat{\lambda}_i^*, X_i) - \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \lambda_\infty^*, X_i) \geq -\epsilon \text{ a.s.}, \end{aligned}$$

and since ϵ is arbitrary,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \lambda_\infty^*, X_i) \geq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \hat{\lambda}_i^*, X_i).$$

Therefore we can conclude that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(S(X_1^{i-1}), \lambda_\infty^*, X_i) \geq \mathcal{V}^* \text{ a.s.}$$

We finish the proof by noticing that since $\mathcal{S} \in \mathcal{S}_\gamma$, then by definition

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N c(S(X_1^{i-1}), X_i) \leq \gamma \text{ a.s.}$$

and since λ_∞^* is non negative, we will get the desired result. \square

3 Proof of Theorem 2

Theorem 2. Assume that $\{X_i\}_{i=-\infty}^{\infty}$ is a γ -feasible process. Then, it is possible to construct a countable set of experts $\{H_{k,h}\}$ for which

$$\lim_{k \rightarrow \infty} \lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_{k,h}^i, X_i) = \mathcal{V}^* \text{ a.s.},$$

where $(y_{k,h}^i, \lambda_{k,h}^i)$ are the predictions made by expert $H_{k,h}$ at round i .

Proof. We start by defining a countable set of experts $\{H_{k,h}\}$ as follow: For $h = 1, 2, \dots$, let $P_h = \{A_{h,j} \mid j = 1, 2, \dots, m_h\}$ be a sequence of finite partitions of \mathcal{X} such that: (i) any cell of P_{h+1} is a subset of a cell of P_h for any h . Namely, P_{h+1} is a refinement of P_h ; (ii) for a set A , if $\text{diam}(A) = \sup_{x,y \in A} \|x - y\|$ denotes the diameter of A , then for any sphere B centered at the origin,

$$\lim_{h \rightarrow \infty} \max_{j: A_{h,j} \cap B \neq \emptyset} \text{diam}(A_{h,j}) = 0.$$

Define the corresponding quantizer $q_h(x) = j$, if $x \in A_{h,j}$. Thus, for any n and X_1^n , we define $Q_h(X_1^n)$ as the sequence $q_h(x_1), \dots, q_h(x_n)$. For expert $H_{k,h}$, we define for $k > 0$, a k -long string of positive integers, denoted by w , the following set,

$$B_{k,h}^{w,(1,n-1)} \triangleq \{x_i \mid k < i < n, Q_h(X_{i-k}^{i-1}) = w\}.$$

We define also

$$h_{k,h}^y(X_1^{n-1}, w) \triangleq \arg \min_{y \in \mathcal{Y}} \left(\max_{\lambda \in \Lambda} \frac{1}{|B_{k,h}^{w,(1,n-1)}|} \sum_{x_i \in B_{k,h}^{w,(1,n-1)}} l_{k,h,n}(y, \lambda, x_i) \right)$$

$$h_{k,h}^\lambda(X_1^{n-1}, w) \triangleq \arg \max_{\lambda \in \Lambda} \left(\min_{y \in \mathcal{Y}} \frac{1}{|B_{k,h}^{w,(1,n-1)}|} \sum_{x_i \in B_{k,h}^{w,(1,n-1)}} l_{k,h,n}(y, \lambda, x_i) \right)$$

for

$$l_{k,h,n}(y, \lambda, x) \triangleq l(y, \lambda, x) + (\|y\|^2 - \|\lambda\|^2) \left(\frac{1}{n} + \frac{1}{h} + \frac{1}{k} \right)$$

and we will set $h_{k,h}^y(X_1^{n-1}, w) = y_0$ and $h_{k,h}^\lambda(X_1^{n-1}, w) = \lambda_0$ for arbitrary $(y_0, \lambda_0) \in \mathcal{Y} \times \Lambda$ if $B_{k,h}^{w,(1,n-1)}$ is empty. Using the above, we define the predictions of $H_{k,h}$ to be:

$$H_{k,h}^y(X_1^{n-1}) = h_{k,h}^y(X_1^{n-1}, Q(X_{n-k}^{n-1})), \quad n = 1, 2, 3, \dots$$

$$H_{k,h}^\lambda(X_1^{n-1}) = h_{k,h}^\lambda(X_1^{n-1}, Q(X_{n-k}^{n-1})), \quad n = 1, 2, 3, \dots$$

We will add two experts: $H_{0,0}$ whose predictions are always (y_0, λ_{\max}) and $H_{-1,-1}$ whose predictions are always $(y_0, 0)$.

Fixing $k, h > 0$ and w , we will define a (random) measure $\mathbb{P}_{j,w}^{(k,h)}$ that is the measure concentrated on the set $B_{k,h}^{w,(0,1-j)}$, defined by

$$\mathbb{P}_{j,w}^{(k,h)}(A) = \frac{\sum_{X_i \in B_{k,h}^{w,(0,1-j)}} 1_A(X_i)}{|B_{k,h}^{w,(0,1-j)}|},$$

where 1_A denotes the indicator function of the set $A \subset \mathcal{X}$. If the above set $B_{k,h}^w$ is empty, then let $P_{j,w}^{(k,h)}(A) = \delta(x')$ be the probability measure concentrated on arbitrary vector $x' \in \mathcal{X}$.

In other words, $\mathbb{P}_{j,w}^{(k,h)}(A)$ is the relative frequency of the the vectors among X_{1-j+k}, \dots, X_0 that fall in the set A . Applying Lemma 1 twice, it is straightforward to prove that for all w , w.p. 1

$$\mathbb{P}_{j,w}^{(k,h)} \rightarrow \begin{cases} \mathbb{P}_{X_0|G_l(X_{-k}^{-1})=w} & \mathbb{P}(G_l(X_{-k}^{-1}) = w) > 0 \\ \delta(x') & \text{otherwise} \end{cases}$$

weakly as $j \rightarrow \infty$, where $\mathbb{P}_{X_0|G_l(X_{-k}^{-1})=w}$ denotes the distribution of the vector X_0 conditioned on the event $G_l(X_{-k}^{-1}) = w$. To see this, let f be a bounded continuous function. Then,

$$\begin{aligned} \int f(x) \mathbb{P}_{j,w}^{(k,h)}(dx) &= \frac{\frac{1}{|1-j+k|} \sum_{X_i \in B_{k,h}^{w,(0,1-j)}} f(X_i)}{\frac{1}{|1-j+k|} |B_{k,h}^{w,(0,1-j)}|} \\ &\rightarrow \frac{\mathbb{E} \left[f(X_0) 1_{G_l(X_{-k}^{-1})=w}(X_0) \right]}{\mathbb{P}(G_l(X_{-k}^{-1}) = w)} = \mathbb{E} [f(X_0) | G_l(X_{-k}^{-1}) = w], \end{aligned}$$

and in case $\mathbb{P}(\|X_{-k}^{-1} - s\| \leq c/l) = 0$, then w.p. 1, $\mathbb{P}_{j,w}^{(k,h)}$ is concentrated on x' for all j . We will denote the limit distribution of $\mathbb{P}_{j,w}^{(k,h)}$ by $\mathbb{P}_w^{*(k,h)}$.

By definition, $(h_{k,h}^y(X_{1-n}^{-1}, w), h_{k,h}^\lambda(X_{1-n}^{-1}, w))$ is the minimax of $l_{n,k,h}$ w.r.t. $\mathbb{P}_{j,w}^{(k,h)}$. The sequence of functions $l_{n,k,h}$ converges uniformly as n approaches ∞ to

$$l_{k,h}(y, \lambda, x) = l(y, \lambda, x) + (\|y\|^2 - \|\lambda\|^2) \left(\frac{1}{h} + \frac{1}{k} \right).$$

Note also that for any fixed \mathbb{Q} , $L_{k,h}(y, \lambda, \mathbb{Q}) = \mathbb{E}_{\mathbb{Q}} [l_{k,h}(y, \lambda, X)]$ is strictly convex in y and strictly concave in λ , and therefore, has a unique saddle-point (see, e.g., [7]). Therefore, since w is arbitrary, and following a Corollary 1 of Lemma 3, we get that a.s.

$$y_{k,h}^n \rightarrow y_{k,h}^* \quad \lambda_{k,h}^n \rightarrow \lambda_{k,h}^*$$

where $(y_{k,h}^*, \lambda_{k,h}^*)$ is the minimax of $L_{k,h}$ w.r.t. $\mathbb{P}_{X_{-k}^{-1}}^{*(k,h)}$. Thus, we can apply Lemma 1 and conclude that as N approaches ∞ ,

$$\frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_{k,h}^i, X_i) \rightarrow \mathbb{E} [l(y_{k,h}^*, \lambda_{k,h}^*, X_0)].$$

a.s.. We now evaluate

$$\lim_{h \rightarrow \infty} \mathbb{E} [l(y_{k,h}^*, \lambda_{k,h}^*, X_0)].$$

Using the properties of the partition P_h (see, e.g., [4, 5]), we get that

$$\mathbb{P}_{X_{-k}^{-1}}^{*(k,h)} \rightarrow \mathbb{P}_{\{X_0|X_{-k}^{-1}\}}$$

weakly as $h \rightarrow \infty$. Moreover, the sequence of functions $l_{k,h}$ converges uniformly as h approaches ∞

$$l_k(y, \lambda, x) = l(y, \lambda, x) + \frac{\|y\|^2 - \|\lambda\|^2}{k}.$$

Note also, that for any fixed \mathbb{Q} , $L_k(y, \lambda, \mathbb{Q}) = \mathbb{E}_{\mathbb{Q}} [l_k(y, \lambda, X)]$ is strictly convex-concave, and therefore, has a unique saddle point. Accordingly, by applying Corollary 1 again, we get that a.s.

$$y_{k,h}^* \rightarrow y_k^* \quad \lambda_{k,h}^* \rightarrow \lambda_k^*$$

where (y_k^*, λ_k^*) is the minimax of L_k w.r.t. $\mathbb{P}_{\{X_0|X_{-k}^{-1}\}}$. Therefore, as h approaches ∞ ,

$$l(y_{k,h}^*, \lambda_{k,h}^*, X_0) \rightarrow l(y_k^*, \lambda_k^*, X_0)$$

a.s.. Thus, by Lebesgue's dominated convergence,

$$\lim_{h \rightarrow \infty} \mathbb{E} [l(y_{k,h}^*, \lambda_{k,h}^*, X_0)] = \mathbb{E} [l(y_k^*, \lambda_k^*, X_0)].$$

Notice that for any $\mathbb{Q} \in \mathbb{P}(\mathcal{X})$, the distance between the saddle point of L_k w.r.t. \mathbb{Q} and the saddle point of L w.r.t. \mathbb{Q} converges to 0 as k approaches ∞ . To see this, notice that

$$\begin{aligned} \min_{y \in \mathcal{Y}} \max_{\lambda \in \Lambda} L(y, \lambda, \mathbb{Q}) - \frac{\|\lambda_{\max}\|^2}{k} &\leq \min_{y \in \mathcal{Y}} \max_{\lambda \in \Lambda} L_k(y, \lambda, \mathbb{Q}) \\ &\leq \min_{y \in \mathcal{Y}} \max_{\lambda \in \Lambda} L(y, \lambda, \mathbb{Q}) + \frac{E}{k} \end{aligned}$$

for some constant E , since \mathcal{Y} is bounded. The last part in our proof will be to show that if $(\hat{y}_k^*, \hat{\lambda}_k^*)$ is the minimax of L w.r.t. $\mathbb{P}_{\{X_0 | X_{-k}^{-1}\}}$, then as k approaches ∞ , $\mathbb{E} \left[l \left(\hat{y}_k^*, \hat{\lambda}_k^*, X_0 \right) \right]$ will converge a.s. to \mathcal{V}^* and so $\mathbb{E} [l(y_k^*, \lambda_k^*, X_0)]$.

To show this, we will use the sub-martingale convergence theorem twice. First, we define Z_k as

$$Z_k \triangleq \min_{y \in \mathcal{Y}(\cdot)} \mathbb{E} \left[\max_{\lambda \in \Lambda(\cdot)} \mathbb{E} [l(y, \lambda, X_0) | X_{-\infty}^{-1}] | X_{-k}^{-1} \right],$$

where the minimum is taken w.r.t. all $\sigma(X_{-k}^{-1})$ -measurable strategies and the maximum is taken w.r.t. all $\sigma(X_{-\infty}^{-1})$ -measurable strategies. Notice that Z_k is a super-martingale. We can see this by using the tower property of conditional expectations,

$$\mathbb{E}[Z_{k+1} | X_{-k}^{-1}] = \mathbb{E} \left[\mathbb{E} [Z_{k+1} | X_{-k-1}^{-1}] | X_{-k}^{-1} \right]$$

and since Z_{k+1} is the optimal choice in \mathcal{Y} w.r.t. to X_{-k-1}^{-1} ,

$$\leq \mathbb{E} \left[\mathbb{E}[Z_k | X_{-k-1}^{-1}] | X_{-k}^{-1} \right] = \mathbb{E}[Z_k | X_{-k}^{-1}] = Z_k.$$

Note also that $\mathbb{E}[Z_k]$ is uniformly bounded. Therefore, we can apply the super-martingale convergence theorem and get that $Z_k \rightarrow Z_\infty$ a.s., where,

$$Z_\infty = \mathbb{E} [l(y_\infty^*, \lambda_\infty^*, X_0) | X_{-\infty}^{-1}] = \mathcal{V}^*,$$

and by using Lebesgue's dominated convergence theorem, also $\mathbb{E}[Z_k] \rightarrow \mathbb{E}[Z_\infty] = \mathcal{V}^*$. Using the same arguments, Z'_k , defined as

$$Z'_k \triangleq \max_{\lambda \in \Lambda(\cdot)} \mathbb{E} \left[\min_{y \in \mathcal{Y}(\cdot)} \mathbb{E} [l(y, \lambda, X_0) | X_{-\infty}^{-1}] | X_{-k}^{-1} \right],$$

where the maximum is taken w.r.t. all $\sigma(X_{-k}^{-1})$ -measurable strategies and the minimum is taken w.r.t. all $\sigma(X_{-\infty}^{-1})$ -measurable strategies, is a sub-martingale that also converges a.s. to Z_∞ and thus $\mathbb{E}[Z'_k] \rightarrow \mathbb{E}[Z_\infty] = \mathcal{V}^*$.

We conclude the proof by noticing that the following relation holds for any k ,

$$\begin{aligned} \mathbb{E}[Z'_k] &= \mathbb{E} \left[\max_{\lambda \in \Lambda(\cdot)} \mathbb{E} \left[\min_{y \in \mathcal{Y}(\cdot)} \mathbb{E} [l(y, \lambda, X_0) | X_{-\infty}^{-1}] | X_{-k}^{-1} \right] \right] \\ &\leq \mathbb{E} \left[\max_{\lambda \in \Lambda(\cdot)} \mathbb{E} \left[\mathbb{E} [l(\hat{y}_k^*, \lambda, X_0) | X_{-\infty}^{-1}] | X_{-k}^{-1} \right] \right] \\ &= \mathbb{E} \left[\max_{\lambda \in \Lambda(\cdot)} \mathbb{E} [l(\hat{y}_k^*, \lambda, X_0) | X_{-k}^{-1}] \right] = \mathbb{E} [l(\hat{y}_k^*, \hat{\lambda}_k^*, X_0)], \end{aligned}$$

and using similar arguments we can show that also

$$\mathbb{E} [l(\hat{y}_k^*, \hat{\lambda}_k^*, X_0)] \leq \mathbb{E}[Z_k],$$

and since both $\mathbb{E}[Z_k]$ and $\mathbb{E}[Z'_k]$ converge to \mathcal{V}^* , we get the desired result. \square

4 Proof of Theorem 3

Before proving the main theorem regarding MHA, we now state and prove the following lemma, which is used in the proof of the main result regarding MHA.

Lemma 4. *Let $\{H_{k,h}\}$ be a countable set of experts as defined in the proof of Theorem 2. Then, the following relation holds a.s.:*

$$\begin{aligned} \inf_{k,h} \limsup_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_i, X_i) &\leq \mathcal{V}^* \\ &\leq \sup_{k,h} \liminf_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_{k,h}^i, X_i), \end{aligned}$$

where (y_i, λ_i) are the predictions of MHA when applied on $\{H_{k,h}\}$.

Proof. Set

$$f(y, \mathbb{Q}) \triangleq \max_{\lambda \in \Lambda} \mathbb{E}_{\mathbb{Q}} [l(y, \lambda, X_0)].$$

We will start from the LHS,

$$\inf_{k,h} \limsup_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_i, X_i), \quad (10)$$

and similarly to Lemma 1, by using the strong law of large numbers we can write

$$\begin{aligned} (10) &= \inf_{k,h} \limsup_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E} [l(y_{k,h}^i, \lambda_i, X_0) \mid X_{1-i}^{-1}] \\ &\leq \inf_{k,h} \limsup_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N f(y_{k,h}^i, \mathbb{P}_{X_0 \mid X_{1-i}^{-1}}) \text{ a.s.} \end{aligned} \quad (11)$$

For fixed $k, h > 0$, from the proof of Theorem (2), $y_{k,h}^i \rightarrow y_{k,h}^*$ a.s. as i approaches ∞ , and from Levy's zero-one law also $\mathbb{P}_{X_0 \mid X_{1-i}^{-1}} \rightarrow \mathbb{P}_{\infty}$ weakly. From Lemma 2 we know that f is continuous, therefore, we can apply Lemma 1 and get that

$$(11) = \inf_{k,h} \mathbb{E} [\mathbb{E} [f(y_{k,h}^*, \mathbb{P}_{\infty})]] \leq \lim_{k \rightarrow \infty} \lim_{h \rightarrow \infty} \mathbb{E} [f(y_{k,h}^*, \mathbb{P}_{\infty})]. \quad (12)$$

From the uniqueness of the saddle point and from the proof of Theorem (2), for fixed $k > 0$,

$$\lim_{h \rightarrow \infty} y_{k,h}^* \rightarrow y_k^*$$

a.s.. Thus, from the continuity of f we get that

$$\lim_{h \rightarrow \infty} f(y_{k,h}^*, \mathbb{P}_{\infty}) \rightarrow f(y_k^*, \mathbb{P}_{\infty})$$

and again by Lebesgue's dominated convergence,

$$(12) = \lim_{k \rightarrow \infty} \mathbb{E} [f(y_k^*, \mathbb{P}_{\infty})] = \lim_{k \rightarrow \infty} \mathbb{E} \left[\max_{\lambda \in \Lambda} \mathbb{E}_{\mathbb{P}_{\infty}} [l(y_k^*, \lambda, X_0)] \right]. \quad (13)$$

Now, from Theorem 2 we know that every accumulation point of the sequence $\{y_k^*\}$ is in the optimal set

$$\arg \min_{y \in \mathcal{Y}} \left(\max_{\lambda \in \Lambda} \mathbb{E}_{\mathbb{P}_{\infty}} [l(y, \lambda, X_0)] \right).$$

Therefore a.s.

$$\lim_{k \rightarrow \infty} \max_{\lambda \in \Lambda} \mathbb{E}_{\mathbb{P}_{\infty}} [l(y_k^*, \lambda, X_0)] \rightarrow \mathbb{E}_{\mathbb{P}_{\infty}} [l(y_{\infty}^*, \lambda_{\infty}^*, X_0)],$$

and using Lebesgue's dominated convergence,

$$(13) = \mathbb{E} [\mathbb{E}_{\mathbb{P}_{\infty}} [l(y_{\infty}^*, \lambda_{\infty}^*, X_0)]] = \mathcal{V}^*.$$

Using similar arguments, we can show the second part of the lemma.

□

We are now ready to state and prove the optimality of MHA.

Theorem 3 (Optimality of MHA). *Let (y_i, λ_i) be the predictions generated by MHA when applied on $\{H_{k,h}\}$ as defined in the proof of Theorem 2. Then, for any γ -feasible process $\{X_i\}_{-\infty}^{\infty}$: MHA is a γ -bounded and γ -universal strategy.*

Proof. We first show that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, X_i) = \mathcal{V}^* \quad a.s. \quad (14)$$

Applying Lemma 5 in [6], we know that the x updates guarantee that for every expert $H_{k,h}$,

$$\frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \leq \frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_i, x_i) + \frac{C_{k,h}}{\sqrt{N}} \quad (15)$$

$$\frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \geq \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_{k,h}^i, x_i) - \frac{C'_{k,h}}{\sqrt{N}}, \quad (16)$$

where $C_{k,h}, C'_{k,h} > 0$ are some constants independent of N . In particular, using Equation (15),

$$\frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \leq \inf_{k,h} \left(\frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_i, x_i) + \frac{C_{k,h}}{\sqrt{N}} \right).$$

Therefore, we get

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \\ & \leq \limsup_{N \rightarrow \infty} \inf_{k,h} \left(\frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_i, x_i) + \frac{C_{k,h}}{\sqrt{N}} \right) \\ & \leq \inf_{k,h} \limsup_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_i, x_i) + \frac{C_{k,h}}{\sqrt{N}} \right) \\ & \leq \inf_{k,h} \limsup_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N l(y_{k,h}^i, \lambda_i, x_i) \right), \end{aligned} \quad (17)$$

where in the last inequality we used the fact that \limsup is sub-additive. Using Lemma (4), we get that

$$\begin{aligned} & (17) \leq \mathcal{V}^* \\ & \leq \sup_{k,h} \liminf_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_{k,h}^i, X_i). \end{aligned} \quad (18)$$

Using similar arguments and using Equation (16) we can show that

$$(18) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i).$$

Summarizing, we have

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \leq \mathcal{V}^* \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i).$$

Therefore, we can conclude that a.s.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, X_i) = \mathcal{V}^*.$$

To show that MHA is indeed a γ -bounded strategy and to shorten the notation, we will denote

$$g(y, \lambda, x) \triangleq \lambda(c(y, x) - \gamma).$$

First, from Equation (16) applied on the expert $H_{0,0}$, we get that:

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_{\max}, x) \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_i, x). \quad (19)$$

Moreover, since l is uniformly continuous, for any given $\epsilon > 0$, there exists $\delta > 0$, such that if $|\lambda' - \lambda| < \delta$, then

$$|l(y, \lambda', x) - l(y, \lambda, x)| < \epsilon$$

for any $y \in \mathcal{Y}$ and $x \in \mathcal{X}$. We also know that

$$\lim_{k \rightarrow \infty} \lim_{h \rightarrow \infty} \lim_{i \rightarrow \infty} \lambda_{k,h}^i = \lambda_\infty^*.$$

Therefore, there exist k_0, h_0, i_0 such that $|\lambda_{k_0, h_0}^i - \lambda_\infty^*| < \delta$ for any $i > i_0$. Since $\lim_{k \rightarrow \infty} \lambda_k^* = \lambda_\infty^*$ there exists k_0 such that $|\lambda_{k_0}^* - \lambda_\infty^*| < \frac{\delta}{3}$. Note that $\lim_{h \rightarrow \infty} \lambda_{k_0, h}^* = \lambda_{k_0}^*$, so there exists h_0 such that $|\lambda_{k_0, h_0}^* - \lambda_{k_0}^*| < \frac{\delta}{3}$. Finally, since $\lim_{i \rightarrow \infty} \lambda_{k_0, i_0}^i = \lambda_{k_0, i_0}^*$, there exists i_0 such that if $i > i_0$, then $|\lambda_{k_0, i_0}^i - \lambda_{k_0, i_0}^*| < \frac{\delta}{3}$. Combining all the above, we get that for k_0, h_0, i_0 if $i > i_0$, then

$$|\lambda_{k_0, h_0}^i - \lambda_\infty^*| < |\lambda_{k_0, h_0}^i - \lambda_{k_0, h_0}^*| + |\lambda_{k_0, h_0}^* - \lambda_{k_0}^*| + |\lambda_{k_0}^* - \lambda_\infty^*| < \delta.$$

Therefore,

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_\infty^*, x_i) - \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \right) \leq \\ & \limsup_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_\infty^*, x_i) - \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_{k_0, h_0}^i, x_i) \right) + \\ & \limsup_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_{k_0, h_0}^i, x_i) - \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \right) \end{aligned} \quad (20)$$

From the uniform continuity we also learn that the first summand is bounded above by ϵ , and from Equation (16), we get that the last summand is bounded above by 0. Thus,

$$(20) \leq \epsilon,$$

and since ϵ is arbitrary, we get that

$$\limsup_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_\infty^*, x_i) - \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) \right) \leq 0.$$

Thus,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_\infty^*, X_i) \leq \mathcal{V}^*,$$

and from Theorem 1 we can conclude that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_\infty^*, X_i) = \mathcal{V}^*.$$

Therefore, we can deduce that

$$\begin{aligned}
& \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_i, x_i) - \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_\infty^*, x_i) = \\
& \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_i, x_i) + \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N -g(y_i, \lambda_\infty^*, x_i) \\
& \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_i, x_i) - \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_\infty^*, x_i) \\
& = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_i, x_i) - \frac{1}{N} \sum_{i=1}^N l(y_i, \lambda_\infty^*, x_i) = 0,
\end{aligned}$$

which results in

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_i, x_i) \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_\infty^*, x_i).$$

Combining the above with Equation (19), we get that

$$\begin{aligned}
& \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_{\max}, x_i) \\
& \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(y_i, \lambda_\infty^*, x_i).
\end{aligned}$$

Since $0 \leq \lambda_\infty^* < \lambda_{\max}$, we get that MHA is γ -bounded. This also implies that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \lambda_i (c(y_i, x_i) - \gamma) \leq 0.$$

Now, if we apply Equation (16) on the expert $H_{-1, -1}$, we get that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \lambda_i (c(y_i, x_i) - \gamma) \geq 0.$$

Thus,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \lambda_i (c(y_i, x_i) - \gamma) = 0,$$

and using Equation (14), we get that MHA is also γ -universal. □

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