Supplementary Material
Accelerated consensus via Min-Sum Splitting

We present, in order, the proofs of Proposition 1, Proposition 2, and Theorem 1.

Proof of Proposition 1. First of all, note that the optimization problem (2) can be casted in the unconstrained formulation of problem (1) upon choosing the hard barrier function: $\phi_{vw}(z,z') := 0$ if $z = z'$ and $\phi_{vw}(z,z') := \infty$ otherwise. With this choice, the minimization inside the definition of the message updates in Algorithm 1 admits the trivial solution $\hat{\mu}^s_{wv} = (\delta - 1)\hat{\xi}^s_{wv} + \delta\hat{\xi}^s_{vw}$. Hence, Algorithm 1 yields the following update for the messages $\hat{\mu}^s = (\hat{\mu}^s_{wv})_{(w,v)\in\mathcal{E}}$:

$$\begin{aligned}\hat{\mu}^s_{wv} = (1-1/\delta)\phi_v - (\delta-1)\hat{\mu}^{s-1}_{wv} + (\delta-1)\sum_{z\in\mathcal{N}(v)}\Gamma_{zv}\hat{\mu}^{s-1}_{zv} \\ + \phi_w - \delta\hat{\mu}^{s-1}_{vw} + \delta\sum_{z\in\mathcal{N}(w)}\Gamma_{zw}\hat{\mu}^{s-1}_{zw}.\end{aligned} \tag{1}$$

In vector form, this update can be written as $\hat{\mu}^s = \hat{k}(\delta) + \widehat{K}(\delta,\Gamma)\hat{\mu}^{s-1}$, where $\hat{k}(\delta)_{wv} := \phi_w + (1-1/\delta)\phi_v$. From the linearity of the message update it follows that if $\phi_v(z) = \frac{1}{2}z^2 - b_v z$ and if we choose the initial messages to be quadratic functions, then the messages at any time $s > 0$ will remain quadratic. Namely, if we adopt the parametrization $\hat{\mu}^0_{vw}(z) = \frac{1}{2}\hat{R}^0_{vw}z^2 - \hat{r}^0_{vw}z$, then we have $\hat{\mu}^s_{vw}(z) = \frac{1}{2}\hat{R}^s_{vw}z^2 - \hat{r}^s_{vw}z$ with the linear and quadratic parameters updated, respectively, according to $\hat{R}^s = (2-1/\delta)\mathbf{1} + \widehat{K}(\delta,\Gamma)\hat{R}^{s-1}$ and $\hat{r}^s = \hat{h}(\delta) + \widehat{K}(\delta,\Gamma)\hat{r}^{s-1}$. The belief function reads $\mu^t_v(z) = \phi_v(z) + \delta\sum_{w\in\mathcal{N}(v)}\Gamma_{wv}\hat{\mu}^t_{wv}(z) = \frac{1}{2}[1+\delta\sum_{w\in\mathcal{N}(v)}\Gamma_{wv}\hat{R}^t_{wv}]z^2 - [b_v + \delta\sum_{w\in\mathcal{N}(v)}\Gamma_{wv}\hat{r}^t_{wv}]z$. As by assumption $1+\delta\sum_{w\in\mathcal{N}(v)}\Gamma_{wv}\hat{R}^t_{wv} > 0$, $x^t_v := \arg\min_{z\in\mathbb{R}}\mu^t_v(z) = \frac{b_v + \delta\sum_{w\in\mathcal{N}(v)}\Gamma_{wv}\hat{r}^t_{wv}}{1+\delta\sum_{w\in\mathcal{N}(v)}\Gamma_{wv}\hat{R}^t_{wv}}$. $\square$

Proof of Proposition 2. Recall from Algorithm 1 the definition of the belief function at time s, i.e., $\mu^s_v := \phi_v + \delta\sum_{w\in\mathcal{N}(v)}\Gamma_{wv}\hat{\mu}^s_{wv}$, and let $\mu^s \in \mathbb{R}^V$ be the vector whose v-th component is μ^s_v. Let $\chi^s \in \mathbb{R}^V$ be the vector whose v-th component is given by the function $\chi^s_v := \phi_v - \delta\sum_{w\in\mathcal{N}(v)}\Gamma_{vw}\hat{\mu}^s_{vw}$. Let $\phi \in \mathbb{R}^V$ be the vector whose v-th component is the function ϕ_v. By taking the summations of update (1) over $w \in \mathcal{N}(v)$ and $v \in \mathcal{N}(w)$, respectively, and by performing the change of variables as prescribed by the definitions of μ^s and χ^s (using that Γ is symmetric), we get that the functions μ^s_v's and χ^s_v's evolve according to the linear system $(\mu^s,\chi^s)^T = K(\delta,\Gamma)(\mu^{s-1},\chi^{s-1})^T$, where the matrix $K(\delta,\Gamma)$ is defined as in (5). From the linearity of the message updates it follows that if we choose the initial messages to be quadratic functions, then the messages at any time $s > 0$ will remain quadratic. Namely, if we adopt the parametrization $\mu^0_v(z) = \frac{1}{2}R^0_v z^2 - r^0_v z$ and $\chi^0_v(z) = \frac{1}{2}Q^0_v z^2 - q^0_v z$, then $\mu^s_v(z) = \frac{1}{2}R^s_v z^2 - r^s_v z$ and $\chi^s_v(z) = \frac{1}{2}Q^s_v z^2 - q^s_v z$, where the linear and quadratic parameters are updated according to

$$\begin{pmatrix} r^s \\ q^s \end{pmatrix} = K(\delta,\Gamma)\begin{pmatrix} r^{s-1} \\ q^{s-1} \end{pmatrix}, \qquad \begin{pmatrix} R^s \\ Q^s \end{pmatrix} = K(\delta,\Gamma)\begin{pmatrix} R^{s-1} \\ Q^{s-1} \end{pmatrix}.$$

If $R^t_v > 0$ the final estimates read $x^t_v := \arg\min_{z\in\mathbb{R}}\mu^t_v(z) = r^t_v/R^t_v$. $\square$

Proof of Theorem 1. We analyze Algorithm 3 with initial conditions $R^0 = Q^0 = \mathbf{1}$ and $r^0 = q^0 = b$. By Proposition 2, the output of this algorithm coincides with the output of Algorithm 2 with initial

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conditions $\hat{R}^0 = \hat{r}^0 = 0$. As $\Gamma = \gamma W$ with $W\mathbf{1} = \mathbf{1}$, we have $\operatorname{diag}(\Gamma\mathbf{1}) = \gamma\operatorname{diag}(\mathbf{1}) = \gamma I$, and the matrix $K(\delta, \Gamma)$ in (5) reads as the matrix K in (6). By the results in [15] (see also [13]), we know that for the choice of γ given in the statement of the theorem the following holds:

1. The matrix K has an eigenvalue 1 and all the remaining $2n-1$ eigenvalues have magnitude strictly less than one.
2. The second largest eigenvalue in magnitude of K is given by the quantity ρ_K defined in the statement of the theorem.

It can be verified that

$$(\mathbf{1}, \mathbf{1})^T K = (\mathbf{1}, \mathbf{1})^T, \qquad K \begin{pmatrix} \mathbf{1} \\ (1-\gamma)\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ (1-\gamma)\mathbf{1} \end{pmatrix}.$$

By Lemma 3 in [24], which is a general version of Theorem 1 in [44], we have $\lim_{t\to\infty} W^t = W^\infty$, where W^∞ is defined as in (6). By taking the limit for t that goes to infinity on the two linear systems that define the message updates in Algorithm 3 we get, respectively,

$$\begin{pmatrix} r^\infty \\ q^\infty \end{pmatrix} = K^\infty \begin{pmatrix} r^0 \\ q^0 \end{pmatrix}, \qquad \begin{pmatrix} R^\infty \\ Q^\infty \end{pmatrix} = K^\infty \begin{pmatrix} R^0 \\ Q^0 \end{pmatrix},$$

which yield $r^\infty = \bar{b}R^\infty$, $q^\infty = (1-\gamma)r^\infty = \bar{b}Q^\infty$, and $R^\infty = \frac{2}{2-\gamma}\mathbf{1}$, $Q^\infty = (1-\gamma)R^\infty$. Hence, we have $r^\infty_v / R^\infty_v = \bar{b}$. The error decomposition

$$x^t_v - \bar{b} = \frac{r^t_v}{R^t_v} - \frac{r^\infty_v}{R^\infty_v} = \frac{r^t_v}{R^t_v} - \frac{r^\infty_v}{R^t_v} + \frac{r^\infty_v}{R^t_v} - \frac{r^\infty_v}{R^\infty_v} = \frac{1}{R^t_v}(r^t_v - r^\infty_v) + \frac{r^\infty_v}{R^t_v R^\infty_v}(R^\infty_v - R^t_v)$$

yields, using that $R^t_v \geq 1$ and $\bar{b} < 1$, by the triangle inequality for the ℓ_2 norm $\|\cdot\|$,

$$\|x^t - \bar{b}\mathbf{1}\| \leq \|r^t - r^\infty\| + \|R^t - R^\infty\| \leq 2\max\{\|r^t - r^\infty\|, \|R^t - R^\infty\|\}.$$

We first bound the term for the quadratic parameters. As

$$\begin{pmatrix} R^t - R^\infty \\ Q^t - Q^\infty \end{pmatrix} = (K - K^\infty)\begin{pmatrix} R^{t-1} \\ Q^{t-1} \end{pmatrix} = (K - K^\infty)\begin{pmatrix} R^{t-1} - R^\infty \\ Q^{t-1} - Q^\infty \end{pmatrix},$$

we have

$$\begin{pmatrix} R^t - R^\infty \\ Q^t - Q^\infty \end{pmatrix} = (K - K^\infty)^t \begin{pmatrix} R^0 - R^\infty \\ Q^0 - Q^\infty \end{pmatrix},$$

from which it follows that

$$\|R^t - R^\infty\| \leq \left\|\begin{pmatrix} R^t - R^\infty \\ Q^t - Q^\infty \end{pmatrix}\right\| \leq \|(K - K^\infty)^t\| \left\|\begin{pmatrix} R^0 - R^\infty \\ Q^0 - Q^\infty \end{pmatrix}\right\|.$$

Given that

$$\left\|\begin{pmatrix} R^0 - R^\infty \\ Q^0 - Q^\infty \end{pmatrix}\right\| = \sqrt{\|\mathbf{1} - R^\infty\|_2^2 + \|\mathbf{1} - Q^\infty\|_2^2},$$

with $\|\mathbf{1} - R^\infty\|_2^2 = \|\mathbf{1} - Q^\infty\|_2^2 = \frac{\gamma^2}{(2-\gamma)^2}n$, we get $\|R^t - R^\infty\| \leq \|(K - K^\infty)^t\| \frac{\gamma}{2-\gamma}\sqrt{2n}$. Proceeding analogously for the linear parameters, we find $\|r^t - r^\infty\| \leq \|(K - K^\infty)^t\| \sqrt{\|r^0 - r^\infty\|_2^2 + \|q^0 - q^\infty\|_2^2}$. We have $\|r^0 - r^\infty\| = \|b - \bar{b}R^\infty\| = \|b - \bar{b}\mathbf{1} + \bar{b}\mathbf{1} - \bar{b}R^\infty\| \leq \|b - \bar{b}\mathbf{1}\| + |\bar{b}|\|\mathbf{1} - R^\infty\|$ so that $\|r^0 - r^\infty\| \leq \sqrt{n} + \frac{\gamma}{(2-\gamma)}\sqrt{n} = \frac{2}{(2-\gamma)}\sqrt{n}$. In the same way we get $\|q^0 - q^\infty\| = \|b - \bar{b}Q^\infty\| \leq \frac{2}{(2-\gamma)}\sqrt{n}$. All together, $\|r^t - r^\infty\| \leq \|(K - K^\infty)^t\| \frac{2}{2-\gamma}\sqrt{2n}$. Finally, as $\gamma < 2$ we obtain $\|x^t - \bar{b}\mathbf{1}\| \leq \frac{4}{2-\gamma}\sqrt{2n}\|(K - K^\infty)^t\|$.

It can be checked that for $z \in [0, 1]$ the following inequalities hold

$$1 - 2z \leq \sqrt{\frac{1 - \sqrt{1 - (1 - z^2)^2}}{1 + \sqrt{1 - (1 - z^2)^2}}} \leq 1 - z.$$

Upon choosing $\rho_W = 1 - z^2$, we recover the bounds stated at the end of Theorem 1. □