Mathematical Arguments

Critical time definition: proofs of Theorem 1 and Proposition 1

Proof of Theorem 1. Let $S = \{T \in \mathbb{R}_+ | \sigma_n(T) = o(n)\}$. S is an interval containing 0 since $\sigma_n(0) = 0$ and, if $T \in S$, then $\forall T' \leq T$, $\sigma_n(T') \leq \sigma_n(T)$ and $T' \in S$. Thus S is of the form $[0, T^c[$ or $[0, T^c]$, and let $T^c = \sup S$ (where $T^c \in \mathbb{R} \cup \{+\infty\}$).

For all time sequences $(T_n)_{n\in\mathbb{N}}$ such that $\limsup_{n\to+\infty} T_n < T^c$, $\exists T < T^c$ and $n' \geq 0$ s.t. , $\forall n \geq n', T_n \leq T$. Hence, by definition of T^c, $\sigma_n(T_n) \leq \sigma_n(T) = o(n)$.

Conversely, if $\sigma_n(T_n) = o(n)$, then $\liminf_{n\to+\infty} T_n \in S$, and $\liminf_{n\to+\infty} T_n \leq T^c$.

Now let $T^{c\prime}$ verify the two constraints of Theorem 1. The first constraint implies that $\forall T < T^{c\prime}$, $T \in S$ and $T \leq T^c$, which leads to $T^{c\prime} \leq T^c$. Moreover, $\forall T < T^c, T \in S$ by definition of T^c, and $T \leq T^{c\prime}$ using the second constraint. As a result, $T^{c\prime} = T^c$ and the critical time is unique. □

Proof of Proposition 1. Let $(T_n)_{n\in\mathbb{N}}$ be such that $\liminf_{n\to+\infty} T_n > T^c$. Then $\exists T > T^c$ and $n' \geq 0$ s.t. $\forall n \geq n'$, $T_n \geq T$. However, $T \notin S$ and $\liminf_{n\to+\infty} \sigma_n(T)/n > 0$, which directly implies that $\liminf_{n\to+\infty} \sigma_n(T_n)/n \geq \liminf_{n\to+\infty} \sigma_n(T)/n > 0$.

Conversely, if $(T_n)_{n\in\mathbb{N}}$ is such that $\liminf_{n\to+\infty} \sigma_n(T_n)/n > 0$, then $\limsup_{n\to+\infty} T_n \notin S$ and $\limsup_{n\to+\infty} T_n \geq T^c$. □

Upper bound on the influence: proofs of Proposition 2 and Corollary 1

Let $\tau_i \in \mathbb{R}_+ \cup \{+\infty\}$ be the infection time of node i, and $\tau_{ij} \in \mathbb{R}_+ \cup \{+\infty\}$ the transmission time from node i to node j. Let $A \subset \mathcal{V}$ be a set of influencers, i.e. nodes that are infected at time 0: $\forall i \in A, \tau_i = 0$. Due to the infection dynamic of CTIC, a node $i \notin A$ is infected when at least one of its neighbors is infected, and the respective ingoing edge transmitted the contagion. We thus have the following equation relating infection times τ_i and τ_{ji} (see for example [1]): $\forall i \notin A$,

$$\tau_i = \min_{j\in\mathcal{V}} \tau_j + \tau_{ji}. \tag{1}$$

Let $X_i(t) = \mathbb{1}_{\{\tau_i < t\}}$ be the infection state of node i at time t. Eq. 1 implies the following equation: $\forall t > 0$ and $i \notin A$,

$$X_i(t) = 1 - \prod_{j\in\mathcal{V}} \left(1 - \mathbb{1}_{\{\tau_j + \tau_{ji} < t\}}\right). \tag{2}$$

We now develop the proofs for Proposition 2 and Corollary 1, which rely on upper bounding the Laplace transform of $\sigma_A(T)$.

Lemma 1. *Define $\rho(s) = \rho(\frac{\mathcal{H}(s)+\mathcal{H}(s)^\top}{2})$. Then, for any A such that $|A| = n_0 < n$, denoting by $\hat{\sigma}_A(s) = \int_0^{+\infty} \sigma_A(t)e^{-st}dt$ the Laplace transform of the expected number of nodes reached by the cascade starting from A at time T:*

$$s\hat{\sigma}_A(s) \leq n_0 + \gamma(s)(n - n_0), \tag{3}$$

where $\gamma(s)$ is the smallest solution in $[0, 1]$ of the following equation:

$$\gamma(s) - 1 + \exp\left(-\rho(s)\gamma(s) - \frac{\rho(s)n_0}{\gamma(s)(n - n_0)}\right) = 0. \tag{4}$$

This result requires two intermediate lemmas: Lemma 2, that proves for $i \in \mathcal{V}$ and $t > 0$ a positive correlation between the events '*node j did not infect node i before time t*' and Lemma 4, that bounds the probability that a given node gets infected before t.

Lemma 2. *$\forall i \notin A$ and $t > 0$, $\{1 - \mathbb{1}_{\{\tau_j + \tau_{ji} < t\}}\}_{j\in\mathcal{V}}$ are positively correlated.*

Proof. Denoting by $\mathcal{Q}_i$ the collection of directed paths in G from the influencers A to node i, we get the following expression for variables $(\tau_i)_{i\in\mathcal{V}}$ [1]:

$$\tau_i = \min_{q\in\mathcal{Q}_i} \sum_{(j,l)\in q} \tau_{jl} \tag{5}$$

Therefore, for all $i \notin A$ and $t > 0$, the functions $f_{ij}(\tau_{kl})_{(k,l)\in\mathcal{E}} = \{1 - \mathbb{1}_{\{\tau_j+\tau_{ji}<t\}}\}_{j\in\mathcal{V}}$ are increasing with the partial order on $(\tau_{kl})_{(k,l)\in\mathcal{E}}$. We will then make use of the FKG inequality [2] :

Lemma 3. *(FKG inequality) Let L be a finite distributive lattice, and μ a nonnegative function on L, such that, for any $(x, y) \in L^2$,*

$$\mu(x \vee y)\mu(x \wedge y) \leq \mu(x)\mu(y) \tag{6}$$

Then, for any non-decreasing function f and g on L

$$\left(\sum_{x\in L} f(x)g(x)\right)\left(\sum_{x\in L} \mu(x)\right) \geq \left(\sum_{x\in L} f(x)\mu(x)\right)\left(\sum_{x\in L} g(x)\mu(x)\right) \tag{7}$$

Due to the independence of $(\tau_{kl})_{(k,l)\in\mathcal{E}}$, the condition in Lemma 3 is met by their joint distribution, which is a product measure on the product space $\mathbb{R}^{\mathcal{E}}$. Lemma 2 is then obtained by applying Lemma 3 to any couple of functions $(f_{ij}, f_{ik})_{(i,j)\in\mathcal{E},(i,k)\in\mathcal{E}}$. More specifically, in our problem setting, L is the set of all $(\tau_{kl})_{(k,l)\in\mathcal{E}}$, $\mu(x) = \prod_{(k,l)\in\mathcal{E}} \mathbb{P}(\tau_{kl} = t_{kl})$ is the joint probability distribution of the τ_{kl} when $x = (t_{kl})_{(k,l)\in\mathcal{E}}$.

□

We then show the following lemma that reveals an implicit inequation satisfied by the X_i.

Lemma 4. *For all $(i, j) \in \mathcal{V}^2$, let p_{ij} be an integrable function such that $\int_0^{+\infty} p_{ij}(t)dt < 1$. For any A such that $|A| = n_0 < n$ and for any $i \notin A$, the probability $\mathbb{E}[X_i(t)]$ that node i will be reached by the contagion originating from A verifies:*

$$\mathbb{E}[X_i(t)] \leq 1 - \exp\Big(- \sum_j (\mathcal{H}_{ji} * \mathbb{E}[X_j])(t)\Big), \tag{8}$$

where $(f * g)(t) = \int f(s)g(t - s)ds$ stands for the convolution of f with g and $\mathcal{H}_{ji}(t) = \frac{\ln(1-\int_0^{+\infty} p_{ji}(s)ds)}{\int_0^{+\infty} p_{ji}(s)ds} p_{ji}(t)$.

Proof. Eq. 2 and the positive correlation of $\{1 - \mathbb{1}_{\{\tau_j+\tau_{ji}<t\}}\}_{j\in\{1,\dots,N\}}$ (Lemma 2) imply that

$$\mathbb{E}[X_i(t)] = 1 - \mathbb{E}[\prod_j (1 - \mathbb{1}_{\{\tau_j+\tau_{ji}<t\}})] \leq 1 - \prod_j \mathbb{E}[1 - \mathbb{1}_{\{\tau_j+\tau_{ji}<t\}}] \tag{9}$$

which leads to

$$\begin{aligned} \mathbb{E}[X_i(t)] \quad &\leq 1 - \textstyle\prod_j \left(1 - \mathbb{E}[\mathbb{1}_{\{\tau_j+\tau_{ji}<t\}}]\right) \\ &= 1 - \textstyle\prod_j \left(1 - \mathbb{E}[\mathbb{E}[X_j(t - \tau_{ji})|\tau_{ji}]]\right), \\ &= 1 - \textstyle\prod_j \left(1 - \int_0^{+\infty} \mathbb{E}[X_j(s)]p_{ji}(t - s)ds\right), \end{aligned} \tag{10}$$

since $\forall i, j \in \mathcal{V}$, τ_j and τ_{ji} are independent and p_{ji} is the probability density of τ_{ji}. Note that, in our setting, we consider that influencer nodes are infected at time 0, and thus are not infectious before $t = 0$. We then linearize the product in Eq. 10:

$$\begin{aligned} \mathbb{E}[X_i(t)] \quad &\leq 1 - \exp\left(\textstyle\sum_j \ln(1 - \int_0^{+\infty} \mathbb{E}[X_j(s)]p_{ji}(t - s)ds)\right) \\ &\leq 1 - \exp\left(\textstyle\sum_j \frac{\ln(1-\int_0^{+\infty} p_{ji}(s)ds)}{\int_0^{+\infty} p_{ji}(s)ds} \int_0^{+\infty} \mathbb{E}[X_j(s)]p_{ji}(t - s)ds\right) \\ &= 1 - \exp\left(-\textstyle\sum_j (\mathcal{H}_{ji} * \mathbb{E}[X_j])(t)\right), \end{aligned} \tag{11}$$

since we have on the one hand, for any $x \in [0,1]$ and $a < 1$, $\ln(1-ax) \geq \ln(1-a)x$ (in Eq. 11, we chose $a = \int_0^{+\infty} p_{ji}(s)ds$ and $x = \frac{\int_0^{+\infty} \mathbb{E}[X_j(s)]p_{ji}(t-s)ds}{\int_0^{+\infty} p_{ji}(s)ds}$), and on the other hand $\mathcal{H}_{ji}(t) = \frac{\ln(1-\int_0^{+\infty} p_{ji}(s)ds)}{\int_0^{+\infty} p_{ji}(s)ds} p_{ji}(t)$ by definition of $\mathcal{H}$. Note that $\frac{\ln(1-\int_0^{+\infty} p_{ji}(s)ds)}{\int_0^{+\infty} p_{ji}(s)ds}$ is approximately 1 when $\int_0^{+\infty} p_{ji}(s)ds$ is close to 0. □

Proof of Lemma 1. From here, Proposition 1 follow from Lemma 4 in the exact same way than, in [3], the proof of Proposition 1 is deduced from Lemma 8. However, we give here the fully detailed proof for sake of completeness.

Let $f_i(s) = \int_0^{+\infty} \mathbb{E}[X_i(t)]se^{-st}dt$, then, using Jensen's inequality, $\forall i \notin A$ and $s \geq 0$,

$$f_i(s) \leq 1 - \exp\Big(- \sum_j \mathcal{L}\mathcal{H}_{ji}(s) f_j(s)\Big), \tag{12}$$

where $\mathcal{L}\mathcal{H}_{ji}(s) = \int_0^{+\infty} \mathcal{H}_{ji}(t)e^{-st}dt$ is the Laplace transform of $\mathcal{H}_{ji}$. Note also that $\forall i \in A, f_i(s) = 1$.

For every $i \in [1...n]$, we define $Z_i = \big(f_i(s)\big)_i$ and the vector $Z = (Z_i)_{i\in[1...n]}$. Using lemma 4 and convexity of exponential function, we have for any $u \in R^n$ such that $\forall i \in A, u_i = 0$ and $\forall i \notin A, u_i \geq 0$,

$$u^\top Z \leq |u|_1 \Big(1 - \textstyle\sum_{i=1}^{n-1} \frac{u_i}{|u|_1} \exp(-(\mathcal{L}\mathcal{H}^\top Z)_i)\Big) \leq |u|_1 \Big(1 - \exp\big(- \frac{Z^\top \mathcal{L}\mathcal{H} u}{|u|_1}\big)\Big) \tag{13}$$

where $|u|_1 = \sum_i |u_i|$ is the L_1-norm of u.

Now taking $u = (1_{i\notin A} Z_i)_i$ and noting that $\forall i, u_i \leq Z_i$, we have

$$\frac{Z^\top Z - n_0}{|Z|_1 - n_0} \leq 1 - \exp\Big(- \frac{Z^\top \mathcal{L}\mathcal{H} Z}{|Z|_1 - n_0}\Big) \leq 1 - \exp\Big(- \frac{\rho(s)(Z^\top Z - n_0)}{|Z|_1 - n_0} - \frac{\rho(s)n_0}{|Z|_1 - n_0}\Big) \tag{14}$$

where $\rho(s) = \rho(\frac{\mathcal{L}\mathcal{H} + \mathcal{L}\mathcal{H}^\top}{2})$. Defining $y = \frac{Z^\top Z - n_0}{|Z|_1 - n_0}$ and $z = |Z|_1 - n_0 = s\hat{\sigma}_A(s) - n_0$, the inequation above rewrites

$$y \leq 1 - \exp\Big(- \rho(s)y - \frac{\rho(s)n_0}{z}\Big) \tag{15}$$

But by Cauchy-Schwarz inequality applied to u, $(n-n_0)(Z^\top Z - n_0) \geq (|Z|_1 - n_0)^2$, which means that $z \leq y(n-n_0)$. We now consider the equation

$$x - 1 + \exp\Big(- \rho(s)x - \frac{\rho(s)n_0}{x(n-n_0)}\Big) = 0 \tag{16}$$

Because the function $f : x \to x - 1 + \exp\big(- \rho(s)x + \frac{\rho(s)n_0}{x(n-n_0)}\big)$ is continuous, verifies $f(1) > 0$ and $\lim_{x\to 0^+} f(x) = -1$, equation 16 admits a solution $\gamma(s)$ in $]0,1[$.

We then prove by contradiction that $z \leq \gamma(s)(n-n_0)$. Let us assume $z > \gamma(s)(n-n_0)$. Then $y \leq 1 - \exp\big(-\rho(s)y - \frac{\rho(s)n_0}{\gamma(s)(n-n_0)}\big)$. But the function $h : x \to x - 1 + \exp\big(-\rho(s)x + \frac{\rho(s)n_0}{\gamma(s)(n-n_0)}\big)$ is convex and verifies $h(0) < 0$ and $h(\gamma(s)) = 0$. Therefore, for any $y > \gamma_1$, $0 = f(\gamma_1) \leq \frac{\gamma(s)}{y} f(y) + (1 - \frac{\gamma(s)}{y}) f(0)$, and therefore $f(y) > 0$. Thus, $y \leq \gamma(s)$. But $z \leq y(n-n_0) \leq \gamma(s)(n-n_0)$ which yields the contradiction. □

Using Lemma 1, we may now prove Proposition 2:

Proof of Proposition 2. $\forall s \geq 0$, $T \geq 0$ and $t \geq 0$, $e^{-st} \geq e^{-sT} 1\!\!1_{\{t<T\}}$, hence, using Lemma 1, $s\hat{\sigma}_A(s) = \sum_i \mathbb{E}[e^{-s\tau_i}] \geq n_0 + (\sigma_A(T) - n_0)e^{-sT}$ which leads to the desired inequality. □

Proof of Corollary 1. Using Eq. 16 and the fact that $1 - e^{-x} \leq x$, we get $\gamma(s) \leq \rho(s)\gamma(s) + \frac{\rho(s)n_0}{\gamma(s)(n-n_0)}$ which rewrites $\gamma(s) \leq \sqrt{\frac{\rho(s)n_0}{(1-\rho(s))(n-n_0)}}$ in the case $\rho(s) < 1$. Therefore,

$$\sigma_A(T) \leq n_0 + \sqrt{n_0(n-n_0)} \min_{\{s \geq 0 | \rho(s) < 1\}} \left(\sqrt{\frac{\rho(s)}{1-\rho(s)}} e^{sT} \right). \tag{17}$$

□

Upper bounds on the critical time: proofs of Corollary 2 and Corollary 3

Proof of Corollary 2. Since e^{-st} is decreasing w.r.t. s, $\mathcal{LH}_{ij}(s)$ is decreasing. Thus, the Perron-Frobenius theorem implies that $\rho(s)$ is decreasing. When $\rho(0) \geq 1$, $\rho^{-1}(1-\epsilon)$ exists and is uniquely defined, and using Corollary 1 and 2, $\sigma_A(T) \leq n_0 + (n-n_0)\gamma(\rho^{-1}(1-\epsilon))e^{\rho^{-1}(1-\epsilon)T} \leq n_0 + \sqrt{\frac{n_0(n-n_0)}{\epsilon}} e^{\rho^{-1}(1-\epsilon)T}$. □

Proof of Corollary 3. If $\limsup_{n\to+\infty} \frac{2\rho^{-1}(1)T_n}{\ln n} < 1$, then $\exists \alpha > 0$ and $n' \geq 0$ s.t. $\forall n \geq n'$, $\rho^{-1}(1)T_n \leq \frac{(1-\alpha)\ln n}{2}$. Furthermore, $\lim_{n\to+\infty} \frac{\rho^{-1}(1-\frac{1}{\ln n})}{\rho^{-1}(1)} = 1$, thus $\exists n'' \geq n'$ s.t. $\forall n \geq n''$, $\rho^{-1}(1-\frac{1}{\ln n}) \leq \frac{1-\alpha/2}{1-\alpha}\rho^{-1}(1)$. Using Corollary 2 with $\epsilon = \frac{1}{\ln n}$, $\sigma_A(T) \leq 1 + \sqrt{\ln n(n-1)} e^{\rho^{-1}(1-\frac{1}{\ln n})T} \leq 1 + \sqrt{\ln n} n^{1-\alpha/4} = o(n)$. □

Application to particular contagion model: proofs of Corollary 4, Proposition 3, Proposition 5 and Corollary 5

Proof of Corollary 4. Taking $\rho(s) = \frac{\lambda}{\lambda+s}\rho_\alpha$, Corollary 1 rewrites

$$\sigma_A(T) \leq n_0 + \sqrt{n_0(n-n_0)} \min_{s \geq 0} \left(\sqrt{\frac{\lambda}{s+\lambda(1-\rho_\alpha)}} e^{sT} \right). \tag{18}$$

The function $f(s) = \sqrt{\frac{\lambda}{s+\lambda(1-\rho_\alpha)}} e^{sT}$ admits a unique minimum in $s_{min} = \frac{1}{2T} + \lambda(\rho_\alpha - 1)$. The minimum for $s \geq 0$ is therefore met for $s = s_{min}$ if $\lambda T(1-\rho_\alpha) < \frac{1}{2}$ and $s = 0$ otherwise. The results follow immediately. □

Proof of Proposition 3. In order to prove Proposition 3, it is sufficient to show that Lemma 4 still holds for the SIR model, with $p_{ij}(t) = \beta \exp(-(\delta+\beta)t)$ for $(i,j) \in \mathcal{E}$. For $i \in \mathcal{V}$, let θ_i be the random removal time of node i. Infection times τ_i are then given by the following expression, where $\mathcal{Q}_i$ is the collection of directed paths in G from the influencers A to node i:

$$\tau_i = \min_{q \in \mathcal{Q}_i} \sum_{(j,l) \in q} \tau_{jl} \mathbb{1}_{\{\tau_{jl} < \theta_j\}} \tag{19}$$

Therefore $\forall i \notin A$ and $t > 0$, the functions $f_{ij}(\tau, \theta) = \{1 - \mathbb{1}_{\{\tau_j + \tau_{ji} < t\}} \mathbb{1}_{\{\tau_{ji} < \theta_j\}}\}_{j \in \mathcal{V}}$ are increasing with respect to the partial order on $\mathbb{R}^{\mathcal{E}} \times \mathbb{R}^{\mathcal{V}}$ defined for any $X^1 = (\tau_1^1, ...\tau_m^1, \theta_1^1 ... \theta_n^1) \in \mathbb{R}^{\mathcal{E}} \times \mathbb{R}^{\mathcal{V}}$ and $X^2 = (\tau_1^2, ...\tau_m^2, \theta_1^2 ... \theta_n^2) \in \mathbb{R}^{\mathcal{E}} \times \mathbb{R}^{\mathcal{V}}$ by:

$$X^1 \geq X^2 \iff \begin{cases} \tau_{ij}^1 \geq \tau_{ij}^2 & \text{for any } (i,j) \in \mathcal{E} \\ \theta_i^1 \leq \theta_j^2 & \text{for any } i \in \mathcal{V} \end{cases}. \tag{20}$$

Variables $(\tau_{ij})_{(i,j)\in\mathcal{E}}$ and $(\theta_i)_{i\in\mathcal{V}}$ being independent, we can still apply FKG inequality (Lemma 3) and deduce the positive correlation, for any $i \notin A$ and $t > 0$, of the random variables $\{1 - \mathbb{1}_{\{\tau_j + \tau_{ji} < t\}} \mathbb{1}_{\{\tau_{ji} < \theta_j\}}\}_{j \in \mathcal{V}}$. We then introduce, for any $(i,j) \in \mathcal{E}$:

$$\overline{\tau_{ji}} = \begin{cases} \tau_{ji} & \text{if } \tau_{ji} < \theta_j \\ +\infty & \text{if } \tau_{ji} \geq \theta_j \end{cases}. \tag{21}$$

It is straightforward that each $\overline{\tau_{ji}}$ is a random variable over $\mathbb{R}_+ \cup \{+\infty\}$ with probability distribution p_{ij}, and that $\overline{\tau_{ji}}$ is independent of τ_j. We also have, for any $i \notin A$, $t > 0$ and $(i,j) \in \mathcal{E}$:

$$\{1 - \mathbb{1}_{\{\tau_j + \tau_{ji} < t\}} \mathbb{1}_{\{\tau_{ji} < \theta_j\}}\} = \{1 - \mathbb{1}_{\{\tau_j + \overline{\tau_{ji}} < t\}}\} \tag{22}$$

Lemma 4 for the SIR case (and therefore Proposition 3 and its subsequent corollaries) are then proved from following the same steps than in the independent transmission events case, except replacing $(\tau_{ji})_{(i,j)\in\mathcal{E}}$ by $(\overline{\tau_{ji}})_{(i,j)\in\mathcal{E}}$ □

Proof of Proposition 5. $\rho(s) = \frac{\beta T_n}{1-e^{-\beta T_n}} \frac{\beta}{\beta+s}(1 - e^{-(\beta+s)T_n})\rho(\mathcal{A}) \leq \frac{\beta^2 T_n \rho(\mathcal{A})}{(1-e^{-\beta T_n})s}$, which implies $\rho^{-1}(1)T_n \leq \frac{(\beta T_n)^2 \rho(\mathcal{A})}{1-e^{-\beta T_n}}$. Let $f(x) = \frac{x^2}{1-e^{-x}}$, f is increasing and $\forall a \geq 0$, $f(x) = a \implies x \geq \sqrt{a}(1 - e^{-\sqrt{a}})$. Hence, if $\limsup_{n\to+\infty} \frac{\beta T_n}{\sqrt{\frac{\ln n}{2\rho(\mathcal{A}_n)}}(1-e^{-\sqrt{\frac{\ln n}{2\rho(\mathcal{A}_n)}}})} < 1$, then $\exists \alpha > 0$ s.t. $\beta T_n \leq (1-\alpha)\sqrt{\frac{\ln n}{2\rho(\mathcal{A}_n)}}(1-e^{-\sqrt{\frac{\ln n}{2\rho(\mathcal{A}_n)}}})$, and the concavity of $1-e^{-x}$ implies that $\beta T_n \leq \sqrt{\frac{(1-\alpha)\ln n}{2\rho(\mathcal{A}_n)}}(1-e^{-\sqrt{\frac{(1-\alpha)\ln n}{2\rho(\mathcal{A}_n)}}})$. Finally, $f(\beta T_n) \leq \frac{(1-\alpha)\ln n}{2\rho(\mathcal{A}_n)}$ and $\frac{2\rho^{-1}(1)T_n}{\ln n} \leq 1 - \alpha$. Applying Corollary 3 proves the desired result. □

Proof of Corollary 5. Taking $\rho(s) = \rho_\alpha e^{-sT_0}$, Corollary 1 rewrites

$$\sigma_A(T) \leq n_0 + \sqrt{n_0(n-n_0)} \min_{s\geq 0} \left(\sqrt{\frac{\rho_\alpha e^{-sT_0}}{1 - \rho_\alpha e^{-sT_0}}} e^{sT} \right). \tag{23}$$

and $s = \frac{1}{T_0}\left(\ln \rho_\alpha - \ln(1 - \frac{T_0}{2T})\right)$ gives

$$\sigma_A(T) \leq n_0 + \sqrt{n_0(n-n_0)} \sqrt{\frac{2T}{T_0} - 1} \left(\frac{\rho_\alpha}{1 - \frac{T_0}{2T}} \right)^{\frac{T}{T_0}}. \tag{24}$$

The final result follows by upper bounding $\left(1 - \frac{T_0}{2T}\right)^{\frac{1}{2} - \frac{T}{T_0}}$ by $\sqrt{e}$ due to the monotonic increase of $x \to (x-1)\ln(1 - \frac{1}{x})$ on $[1, +\infty[$ and its limit when $x \to +\infty$. □

Additional references

[1] Nan Du, Le Song, Manuel Gomez-Rodriguez, and Hongyuan Zha. Scalable influence estimation in continuous-time diffusion networks. In *Advances in Neural Information Processing Systems*, pages 3147–3155, 2013.

[2] Cees M Fortuin, Pieter W Kasteleyn, and Jean Ginibre. Correlation inequalities on some partially ordered sets. *Communications in Mathematical Physics*, 22(2):89–103, 1971.

[3] Remi Lemonnier, Kevin Scaman, and Nicolas Vayatis. Tight bounds for influence in diffusion networks and application to bond percolation and epidemiology. In *Advances in Neural Information Processing Systems*, pages 846–854, 2014.