
SGD Algorithms based on Incomplete U -statistics: Large-Scale Minimization of Empirical Risk Supplementary Material

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1 Proof of Proposition 2

We follow the proof of [1] to derive bounds. We highlight the fact that since the loss function H is l -smooth, $\hat{g}_n(\theta_t)$ and $\hat{g}_n'(\theta_t)$ are l -Lipschitz. We introduce the sequence $\tilde{\gamma}_t = \gamma_t(1 - l\gamma_t)$. In all generality we will denote by $g_t(\theta)$ an unbiased estimator of the gradient at iteration t , l -Lipschitz in θ . We study the recursion $\theta_{t+1} = \theta_t - \gamma_t g_t(\theta_t)$.

We will make use of the two following classical inequalities from convex analysis (see [7]) :

$$\hat{L}_n(\theta_1) - \hat{L}_n(\theta_2) \leq \nabla \hat{L}_n(\theta_1)^T(x - y) - \frac{\alpha}{2} \|\theta_1 - \theta_2\|^2 \quad (1)$$

$$\frac{1}{l} \|g_t(\theta_1) - g_t(\theta_2)\|^2 \leq (g_t(\theta_1) - g_t(\theta_2))^T(\theta_1 - \theta_2) \quad (2)$$

As mentioned previously the analysis we proposed can easily be extended to a more general setting as in [1]. We now begin the proof of the proposition :

$$\|\theta_{t+1} - \theta_n^*\|^2 = \|\theta_t - \theta_n^*\|^2 - 2\gamma_t g_t(\theta_t)^T(\theta_t - \theta_n^*) + \gamma_t^2 \|g_t(\theta_t)\|^2$$

Using (2) we get

$$\begin{aligned} \|g_t(\theta_t)\|^2 &\leq 2(\|g_t(\theta_t) - g_t(\theta_n^*)\|^2 + \|g_t(\theta_n^*)\|^2) \\ &\leq 2l(g_t(\theta_t) - g_t(\theta_n^*))^T(\theta_t - \theta_n^*) + 2\|g_t(\theta_n^*)\|^2 \end{aligned} \quad (3)$$

which together with $\mathbb{E}_n[g_t(\theta_t)|\theta_t] = \hat{g}_n(\theta_t)$ gives

$$\mathbb{E}_n[\|\theta_{t+1} - \theta_n^*\|^2|\theta_t] \leq \|\theta_t - \theta_n^*\|^2 - 2\tilde{\gamma}_t \hat{g}_n(\theta_t)^T(\theta_t - \theta_n^*) + 2\gamma_t^2 \|g_t(\theta_n^*)\|^2$$

For the sake of simplicity we assume $(1 - l\gamma_t) > 0 \forall t$ (which is eventually true since the sequence $(\gamma_t)_{t \geq 0}$ goes to 0 as t goes to infinity). Let $a_t = \mathbb{E}_n[\|\theta_t - \theta_n^*\|^2]$, $\sigma_n^2(\theta_n^*)$ the variance (conditionally upon the data) of $g_t(\theta_n^*)$. Using (1) and taking the expectation we get the following recursion :

$$\begin{aligned} a_{t+1} &\leq a_t (1 - 2\alpha\tilde{\gamma}_t) + 2\gamma_t^2 \sigma_n^2(\theta_n^*) \\ &\leq a_1 \prod_{j=1}^t (1 - 2\alpha\tilde{\gamma}_j) + 2\sigma_n^2(\theta_n^*) \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) \end{aligned} \quad (4)$$

with the convention $\prod_{k=t+1}^t (1 - 2\alpha\tilde{\gamma}_k) = 1$. Using $1 + x \leq e^x$ we get the following upper bound :

$$\prod_{j=1}^t (1 - 2\alpha\tilde{\gamma}_j) \leq \exp(-2\alpha \sum_{j=1}^t \gamma_j) \exp(2\alpha l \sum_{j=1}^t \gamma_j^2)$$

We now need to distinguish two cases :

1.1 Case $\beta = 1$

If $\beta = 1$ we have :

1. $\log(t+1) - \log(j+1) \leq \sum_{k=j+1}^t \frac{1}{k}$
2. $\exp(2\alpha l \sum_{k=1}^t \frac{1}{k^2}) \leq \exp(4\alpha l)$
3. $\exp(2\alpha l \sum_{k=j+1}^t \frac{1}{k^2}) \leq \exp(\frac{2\alpha l}{j}) \leq \exp(2\alpha l)$

Under the assumption $2\alpha\gamma_1 > 1$:

$$\begin{aligned} a_{t+1} &\leq \frac{a_1}{(t+1)^{2\alpha\gamma_1}} \exp(4\alpha l \gamma_1^2) + 2\sigma_n^2(\theta_n^*) \exp(2\alpha l \gamma_1^2) \gamma_1^2 \sum_{j=1}^t \frac{(j+1)^{2\alpha\gamma_1}}{j^2} \frac{1}{(t+1)^{2\alpha\gamma_1}} \\ &\leq \frac{a_1}{(t+1)^{2\alpha\gamma_1}} \exp(4\alpha l \gamma_1^2) + \frac{2^{\alpha\gamma_1} 2\sigma_n^2(\theta_n^*) \exp(2\alpha l \gamma_1^2) \gamma_1^2}{(2\alpha\gamma_1 - 1)(t+1)} \end{aligned}$$

which gives the result.

1.2 Case $\beta < 1$

If $\beta < 1$, let t_0 be a positive index, by splitting the sum in two parts we get :

$$\begin{aligned} \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) &= \sum_{j=1}^{t_0} \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) + \sum_{j=t_0+1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) \\ &\leq \prod_{k=t_0+1}^t (1 - 2\alpha\tilde{\gamma}_k) \sum_{j=1}^{t_0} \gamma_j^2 + \gamma_{t_0} \sum_{j=t_0+1}^t \gamma_j \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) \end{aligned}$$

where we used that the sequence $(\gamma_t)_{t \geq 1}$ is decreasing. Since $\gamma_j = \frac{1-(1-2\alpha\tilde{\gamma}_j)}{2\alpha} + l\gamma_j^2$ we have :

$$\begin{aligned} \sum_{j=t_0+1}^t \gamma_j \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) &= \frac{1}{2\alpha} \sum_{j=t_0+1}^t \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) - \prod_{k=j}^t (1 - 2\alpha\tilde{\gamma}_k) \\ &\quad + l \sum_{j=t_0+1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) \leq \frac{1}{2\alpha} + l \sum_{j=t_0+1}^t \gamma_j^2 \end{aligned}$$

which leads to

$$\begin{aligned} \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) &\leq \exp(-2\alpha \sum_{j=t_0+1}^t \gamma_j) \exp(2\alpha l \sum_{j=t_0+1}^t \gamma_j^2) \sum_{j=1}^{t_0} \gamma_j^2 \\ &\quad + \frac{\gamma_{t_0}}{2\alpha} + \gamma_{t_0} l \sum_{j=t_0+1}^t \gamma_j^2 \end{aligned}$$

Taking $t_0 \sim \frac{t}{2}$ and using the integral test for convergence :

1. $\sum_{j=t_0+1}^t \gamma_j = \gamma_1 \sum_{j=t_0+1}^t \frac{1}{j^\beta} \geq \gamma_1 \frac{(t+1)^{1-\beta} - (t_0+1)^{1-\beta}}{1-\beta} \geq \gamma_1 \frac{(t+1)^{1-\beta}}{2(1-\beta)}$
2. $\sum_{j=t_0+1}^t \gamma_j^2 = \gamma_1^2 \sum_{j=t_0+1}^t \frac{1}{j^{2\beta}} \leq \gamma_1^2 \sum_{j=2}^{+\infty} \frac{1}{j^{2\beta}} \leq \frac{\gamma_1^2}{2\beta-1}$
3. $\sum_{j=1}^{t_0} \gamma_j^2 \leq \gamma_1^2 (1 + \frac{1}{2\beta-1}) = \frac{2\beta}{2\beta-1} \gamma_1^2$

gives the following bound :

$$\begin{aligned}
a_{t+1} &\leq a_1 \exp(-2\alpha\gamma_1 \frac{(t+1)^{1-\beta}}{2(1-\beta)}) \exp(2\alpha l \frac{\gamma_1^2}{2\beta-1}) \\
&\quad + 2\sigma_n^2(\theta_n^*) (\exp(-2\alpha \frac{(t+1)^{1-\beta}}{2(1-\beta)}) \exp(2\alpha l \frac{\gamma_1^2}{2\beta-1}) \frac{2\beta}{2\beta-1} \gamma_1^2 \\
&\quad + \frac{2\beta\gamma_1}{2\alpha t^\beta} + \frac{\gamma_1 2^\beta}{t^\beta} \frac{2l\beta}{2\beta-1} \gamma_1^2) = \sigma_n^2(\theta_n^*) \frac{\gamma_1 2^\beta}{t^\beta} (\frac{1}{2\alpha} + \frac{l\gamma_1^2}{2\beta-1}) + o(\frac{1}{t^\beta})
\end{aligned}$$

which concludes the proof.

2 Proof of Theorem 1

We recall that $\Gamma = \nabla^2 \widehat{L}_{\mathbf{n}}(\theta_n^*)$, $\Sigma_n(\theta_n^*) = \mathbb{E}_{\mathbf{n}}[g_t(\theta_n^*)g_t(\theta_n^*)^T]$ and Σ_n^* is the solution of Lyapunov's equation :

$$\Gamma \Sigma_n^* + \Sigma_n^* \Gamma - \eta \Sigma_n^* = \Sigma_n(\theta_n^*), \quad (5)$$

Using classical results from stochastic approximation theory (see [4, 5, 8] for instance) , we first show that under our assumptions:

$$\sqrt{1/\gamma_t} (\theta_t - \theta_n^*) \Rightarrow \mathcal{N}(0, \Sigma_n^*),$$

The asymptotic behavior of $1/\gamma_t (\widehat{L}_{\mathbf{n}}(\theta_t) - \widehat{L}_{\mathbf{n}}(\theta^*))$ is therefore a consequence of the second order delta method. We now turn to the second part of proposition 3 and follow the analysis of [3]: We have

$$\begin{aligned}
\mathbb{E}_{\mathbf{n}}[U^T (\Sigma_n^*)^{1/2} \Gamma (\Sigma_n^*)^{1/2} U] &= \mathbb{E}_{\mathbf{n}}[Tr(\Gamma^{1/2} (\Sigma_n^*)^{1/2} U U^T (\Sigma_n^*)^{1/2} \Gamma^{1/2})] \\
&= Tr(\Gamma^{1/2} (\Sigma_n^*)^{1/2} \mathbb{E}_{\mathbf{n}}[U U^T] (\Sigma_n^*)^{1/2} \Gamma^{1/2}) \\
&= Tr(\Gamma^{1/2} (\Sigma_n^*) \Gamma^{1/2}) = Tr(\Gamma \Sigma_n^*) \\
&= \frac{1}{2} Tr(\Sigma_n(\theta_n^*)) = \frac{1}{2} \sigma_n^2(\theta_n^*)
\end{aligned}$$

where we used the linearity of the trace, the linearity of the expectation, the dominated convergence theorem (to arrange the different terms) and Lyapunov's equation to conclude.

3 Proof of Theorem 2

We prove a more general result and apply it to our specific setting. We consider the recursion defined in the proof of Proposition 2 and keep the same notations.

Theorem. *Let θ_t be the sequence generated by SGD and define $\sigma^2 = \mathbb{E}[\sigma_{\mathbf{n}}^2(g(\theta_n^*))]$. Assume that $\{L(\cdot; \theta) : \theta \in \Theta\}$ is a VC major class class of finite VC dimension V s.t*

$$\mathcal{M}_{\Theta} = \sup_{\theta \in \Theta, (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(K)}) \in \prod_{k=1}^K \mathcal{X}_k^{d_k}} |H(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(K)}; \theta)| < +\infty, \quad (6)$$

If the step size satisfies the condition of Proposition 2, we have:

$$\forall \mathbf{n} \in \mathbb{N}^{*K}, \quad \mathbb{E}[|L(\theta_t) - L(\theta^*)|] \leq \frac{C\sigma^2}{t^\beta} + 2\mathcal{M}_{\Theta} \left\{ 2\sqrt{\frac{2V \log(1+\kappa)}{\kappa}} \right\}.$$

For any $\delta \in (0, 1)$, we also have with probability at least $1 - \delta$: $\forall \mathbf{n} \in \mathbb{N}^{*K}$,

$$|L(\theta_t) - L(\theta^*)| \leq \left(\frac{C\sigma^2}{t^\beta} + \sqrt{\frac{D_\beta \log(2/\delta)}{t^\beta}} \right) + 2\mathcal{M}_{\Theta} \left\{ 2\sqrt{\frac{2V \log(1+\kappa)}{\kappa}} + \sqrt{\frac{\log(4/\delta)}{\kappa}} \right\}. \quad (7)$$

for some constant D_β depending on the parameters l, α, γ_1, a_1 and where $C = C_1$ if $\beta < 1$ and C_2 otherwise.

Proof. For the sake of simplicity, we place ourselves in the special case where Θ is compact, but tedious calculations would lead to similar results under less restrictive assumptions. We therefore introduce the quantities M and M_1 that satisfy $\|g_t(\theta_n^*)\|^2 \leq M^2$ and $\|\theta_t - \theta_n^*\| \leq M_1^2$. We now turn to the proof of the result :

$$L(\theta_t) - L(\theta^*) \leq 2 \sup_{\theta \in \Theta} |\widehat{L}_{\mathbf{n}}(\theta) - L(\theta)| + \widehat{L}_{\mathbf{n}}(\theta_t) - \widehat{L}_{\mathbf{n}}(\theta_n^*).$$

Taking a union bound we directly get :

$$\begin{aligned} \mathbb{P} \left(L(\theta_t) - L(\theta^*) \geq \frac{l}{2} \frac{\sigma^2}{t^\beta} C + \epsilon \right) &\leq \underbrace{\mathbb{P} \left(|\widehat{L}_{\mathbf{n}}(\theta_t) - \widehat{L}_{\mathbf{n}}(\theta_n^*)| \geq \frac{l}{2} \frac{\sigma^2}{t^\beta} C + \frac{\epsilon}{2} \right)}_{\mathcal{P}_1} \\ &\quad + \underbrace{\mathbb{P} \left(\sup_{\theta \in \Theta} |\widehat{L}_{\mathbf{n}}(\theta) - L(\theta)| \geq \frac{\epsilon}{4} \right)}_{\mathcal{P}_2} \end{aligned}$$

The analysis of \mathcal{P}_2 is classical and we refer to [2] to obtain that for all $\delta \in (0, 1)$, we have with probability at least $1 - \delta$,

$$\sup_{\theta \in \Theta} |\widehat{L}_{\mathbf{n}}(\theta) - L(\theta)| \leq \mathcal{M}_{\mathcal{H}} \left\{ 2\sqrt{\frac{2V \log(1 + \kappa)}{\kappa}} + \sqrt{\frac{\log(1/\delta)}{\kappa}} \right\}. \quad (8)$$

We now focus on the second term.

Using $\widehat{L}_{\mathbf{n}}(\theta) - \widehat{L}_{\mathbf{n}}(\theta_n^*) \leq \frac{l}{2} \|\theta - \theta_n^*\|^2$ (see [6] for instance) we have :

$$\mathbb{P} \left(|\widehat{L}_{\mathbf{n}}(\theta_t) - \widehat{L}_{\mathbf{n}}(\theta_n^*)| - \frac{l}{2} \frac{\sigma^2}{t^\beta} C \geq \frac{\epsilon}{2} \right) \leq \mathbb{P} \left(\|\theta_t - \theta_n^*\|^2 - \frac{\sigma^2}{t^\beta} C \geq \frac{\epsilon}{l} \right)$$

Applying the recursion we get:

$$\begin{aligned} \|\theta_{t+1} - \theta_n^*\|^2 &= \|\theta_t - \theta_n^*\|^2 - 2\gamma_t g_t(\theta_t)^T (\theta_t - \theta_n^*) + \gamma_t^2 \|g_t(\theta_t)\|^2 \\ &= \|\theta_t - \theta_n^*\|^2 - 2\gamma_t (g_t(\theta_t) - \nabla \widehat{L}_{\mathbf{n}}(\theta_t))^T (\theta_t - \theta_n^*) - 2\gamma_t \nabla \widehat{L}_{\mathbf{n}}(\theta_t)^T (\theta_t - \theta_n^*) + \gamma_t^2 \|g_t(\theta_t)\|^2 \end{aligned}$$

and using (3) :

$$\begin{aligned} \|g_t(\theta_t)\|^2 &\leq 2l(g_t(\theta_t) - g_t(\theta_n^*))^T (\theta_t - \theta_n^*) + 2\|g_t(\theta_n^*)\|^2 \\ &= 2l(g_t(\theta_t) - \nabla \widehat{L}_{\mathbf{n}}(\theta_t) - g_t(\theta_n^*))^T (\theta_t - \theta_n^*) + 2l(\nabla \widehat{L}_{\mathbf{n}}(\theta_t))^T (\theta_t - \theta_n^*) + 2\|g_t(\theta_n^*)\|^2 \end{aligned}$$

which with $\tilde{a}_t := \|\theta_{t+1} - \theta_n^*\|^2$ gives

$$\begin{aligned} \tilde{a}_{t+1} &\leq \tilde{a}_t (1 - 2\alpha\tilde{\gamma}_t) + 2\gamma_t^2 \sigma_n^2(\theta_n^*) - 2\tilde{\gamma}_t (g_t(\theta_t) - \nabla \widehat{L}_{\mathbf{n}}(\theta_t))^T (\theta_t - \theta_n^*) \\ &\quad - 2\gamma_t^2 l(g_t(\theta_n^*))^T (\theta_t - \theta_n^*) + 2\gamma_t^2 (\|g_t(\theta_n^*)\|^2 - \sigma_n^2(\theta_n^*)). \end{aligned}$$

An immediate recursion leads to

$$\begin{aligned} \tilde{a}_{t+1} &\leq \tilde{a}_1 \prod_{j=1}^t (1 - 2\alpha\tilde{\gamma}_j) + 2\sigma_n^2(\theta_n^*) \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) \\ &\quad + 2 \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) (\|g_t(\theta_n^*)\|^2 - \sigma_n^2(\theta_n^*)) \\ &\quad - 2 \sum_{j=1}^t \tilde{\gamma}_j \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) (g_t(\theta_t) - \nabla \widehat{L}_{\mathbf{n}}(\theta_t))^T (\theta_t - \theta_n^*) \\ &\quad - 2l \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) g_t(\theta_n^*)^T (\theta_t - \theta_n^*) \end{aligned}$$

The first two terms are analyzed in Section 1. We turn now to the tree remaining terms and we introduce the following quantities :

1. $S_{1,t} = 2 \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) (\|g_t(\theta_n^*)\|^2 - \sigma_n^2(\theta_n^*))$
2. $S_{2,t} = 2 \sum_{j=1}^t \tilde{\gamma}_j \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) (g_t(\theta_t) - \nabla \hat{l}_n(\theta_t))^T (\theta_t - \theta_n^*)$
3. $S_{3,t} = 2l \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) g_t(\theta_n^*)^T (\theta_t - \theta_n^*)$

Placing ourself under the conditional probability and applying a union bound yields

$$\begin{aligned} \mathbb{P}_n \left(\|\theta_{t+1} - \theta_n^*\|^2 \geq \tilde{a}_1 \prod_{j=1}^t (1 - 2\alpha\tilde{\gamma}_j) + 2\sigma_n^2(\theta_n^*) \sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) + \epsilon \right) \\ \leq \mathbb{P}_n(S_{1,t} \geq \frac{\epsilon}{3}) + \mathbb{P}_n(S_{2,t} \geq \frac{\epsilon}{3}) + \mathbb{P}_n(S_{3,t} \geq \frac{\epsilon}{3}) \end{aligned}$$

Under our assumptions, we have $\|g_t(\theta_n^*)\|^2 \leq M^2$, $|g_t(\theta_n^*)^T(\theta_t - \theta_n^*)| \leq MM_1$ and $|(g_t(\theta_t) - \nabla \hat{L}_n(\theta_t))^T(\theta_t - \theta_n^*)| \leq (2lM_1 + M)M_1$. Applying Azuma's inequality yields the following bounds:

$$\begin{aligned} \mathbb{P}_n(S_{1,t} \geq \epsilon) &\leq \exp \left(\frac{-\epsilon^2}{4M^4 \sum_{j=1}^t \gamma_j^4 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k)^2} \right) \\ \mathbb{P}_n(S_{2,t} \geq \epsilon) &\leq \exp \left(\frac{-\epsilon^2}{8M_1^2(2lM_1 + M)^2 \sum_{j=1}^t \tilde{\gamma}_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k)^2} \right) \\ \mathbb{P}_n(S_{3,t} \geq \epsilon) &\leq \exp \left(\frac{-\epsilon^2}{8M^2M_1^2 \sum_{j=1}^t \gamma_j^4 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k)^2} \right) \end{aligned}$$

We thus need to bound the sums $\sum_{j=1}^t \tilde{\gamma}_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k)^2$ and $\sum_{j=1}^t \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k)^2$.

3.1 Case $\beta < 1$

We have for $\beta < 1$:

$$\begin{aligned} \sum_{t=1}^T \gamma_j^4 \prod_{k=j+1}^T (1 - 2\alpha\tilde{\gamma}_k)^2 &\leq \prod_{j=t_0+1}^T (1 - 2\alpha\tilde{\gamma}_j)^2 \sum_{j=1}^{t_0} \gamma_j^4 + \gamma_{t_0}^2 \sum_{j=t_0+1}^T \gamma_j^2 \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k)^2 \\ &\leq \exp(-4\alpha \sum_{j=t_0+1}^T \gamma_j) \exp(4\alpha l \sum_{j=t_0+1}^T \gamma_j^2) \sum_{j=1}^{t_0} \gamma_j^4 \\ &\quad + \gamma_{t_0}^2 \left(\frac{1}{2\alpha} + l \sum_{j=t_0+1}^T \gamma_j^2 \right)^2 \\ &\leq \exp(-4\alpha\gamma_1 \frac{(T+1)^{(1-\beta)}}{2(1-\beta)}) \exp(\frac{4\alpha l \gamma_1^2}{2\beta-1}) \frac{4\beta}{4\beta-1} \gamma_1^4 \\ &\quad + \frac{\gamma_1^2 2^{2\beta}}{T^{2\beta}} \left(\frac{1}{2\alpha} + \frac{2l\beta}{2\beta-1} \right)^2 \\ &= \frac{\gamma_1^2 2^{2\beta}}{T^{2\beta}} \left(\frac{1}{2\alpha} + \frac{2l\beta}{2\beta-1} \right)^2 + o(\frac{1}{T^{2\beta}}) \end{aligned}$$

where we used $\sum_{j=1}^T x_j^2 \leq (\sum_{j=1}^T x_j)^2$ for $x_j = \gamma_j \prod_{k=j+1}^t (1 - 2\alpha\tilde{\gamma}_k) \geq 0$ for the last inequality. We now analyze the second sum :

$$\begin{aligned}
\sum_{j=1}^T \tilde{\gamma}_j^2 \prod_{k=j+1}^T (1 - 2\alpha\tilde{\gamma}_k)^2 &\leq \prod_{j=t_0+1}^T (1 - 2\alpha\tilde{\gamma}_j)^2 \sum_{j=1}^{t_0} \tilde{\gamma}_j^2 + \tilde{\gamma}_{t_0} \sum_{j=t_0+1}^T \tilde{\gamma}_j \prod_{k=j+1}^T (1 - 2\alpha\tilde{\gamma}_k)^2 \\
&\leq \exp(-4\alpha \sum_{j=t_0+1}^T \gamma_j) \exp(4\alpha l \sum_{j=t_0+1}^T \gamma_j^2) \sum_{j=1}^{t_0} \tilde{\gamma}_j^2 \\
&\quad + \tilde{\gamma}_{t_0} \sum_{j=t_0+1}^T \frac{1 - (1 - 2\alpha\tilde{\gamma}_j)}{2\alpha} \prod_{k=j+1}^T (1 - 2\alpha\tilde{\gamma}_k) \\
&\leq \exp(-4\alpha \sum_{j=t_0+1}^T \gamma_j) \exp(4\alpha l \sum_{j=t_0+1}^T \gamma_j^2) \sum_{j=1}^{t_0} \tilde{\gamma}_j^2 + \frac{\tilde{\gamma}_{t_0}}{2\alpha} \\
&\leq \exp(-4\alpha\gamma_1 \frac{(T+1)^{(1-\beta)}}{2(1-\beta)}) \exp(\frac{4\alpha l \gamma_1^2}{2\beta-1}) \frac{8\beta}{2\beta-1} \gamma_1^2 + \frac{\gamma_1 2^\beta}{\alpha T^\beta} \\
&= \frac{\gamma_1 2^\beta}{\alpha T^\beta} + o(\frac{1}{T^\beta})
\end{aligned}$$

which concludes this case.

3.2 Case $\beta = 1$

We have:

$$\begin{aligned}
\sum_{t=1}^T \gamma_j^4 \prod_{k=j+1}^T (1 - 2\alpha\tilde{\gamma}_k)^2 &\leq \sum_{t=1}^T \gamma_j^4 \exp(-4\alpha \sum_{j=t+1}^T \gamma_j) \exp(4\alpha l \sum_{j=t+1}^T \gamma_j^2) \\
&\leq \gamma_1^4 \exp(8\alpha l \gamma_1^2) \sum_{t=1}^T \frac{(j+1)^{4\alpha\gamma_1}}{j^4 (T+1)^{4\alpha\gamma_1}} \\
&\leq \frac{\gamma_1^4 \exp(8\alpha l \gamma_1^2) 2^{4\alpha\gamma_1}}{(T+1)^{4\alpha\gamma_1}} \sum_{t=1}^T \frac{1}{j^{4-4\alpha\gamma_1}}
\end{aligned}$$

We thus need to distinguish several cases:

1. If $2 < 4\alpha\gamma_1 < 3$ then

$$\sum_{t=1}^T \frac{1}{j^{4-4\alpha\gamma_1}} \leq 1 + \frac{1}{3 - 4\alpha\gamma_1} = \frac{4 - 4\alpha\gamma_1}{3 - 4\alpha\gamma_1}$$

so

$$\sum_{t=1}^T \gamma_j^4 \prod_{k=j+1}^T (1 - 2\alpha\tilde{\gamma}_k)^2 \leq \frac{\gamma_1^4 \exp(8\alpha l \gamma_1^2) 2^{4\alpha\gamma_1}}{(T+1)^{4\alpha\gamma_1}} \frac{4 - 4\alpha\gamma_1}{3 - 4\alpha\gamma_1}$$

2. $4\alpha\gamma_1 = 3$ then

$$\sum_{t=1}^T \frac{1}{j^{4-4\alpha\gamma_1}} \leq 1 + \log(T)$$

so

$$\sum_{t=1}^T \gamma_j^4 \prod_{k=j+1}^T (1 - 2\alpha\tilde{\gamma}_k)^2 \leq \gamma_1^4 \exp(8\alpha l \gamma_1^2) 2^3 \frac{1 + \log(T)}{(T+1)^3}$$

3. $4\alpha\gamma_1 > 3$ then

$$\sum_{t=1}^T \frac{1}{j^{4-4\alpha\gamma_1}} \leq \frac{(T+1)^{4\alpha\gamma_1-3}}{4\alpha\gamma_1-3}$$

so

$$\sum_{t=1}^T \gamma_j^4 \prod_{k=j+1}^T (1-2\alpha\tilde{\gamma}_k)^2 \leq \frac{\gamma_1^4 \exp(8\alpha l \gamma_1^2) 2^{4\alpha\gamma_1}}{(4\alpha\gamma_1-3)(T+1)^3}$$

Consider $\gamma_1 < \frac{1}{2L}$, then $\tilde{\gamma}_j < \frac{1}{2}\gamma_j$, and

$$\begin{aligned} \sum_{j=1}^t \tilde{\gamma}_j^2 \prod_{k=j+1}^T (1-2\alpha\tilde{\gamma}_k)^2 &\leq 4\gamma_1^2 \sum_{j=1}^t \frac{1}{j^2} \exp(-4\alpha \sum_{j=t+1}^T \gamma_j) \exp(4\alpha l \sum_{j=t+1}^T \gamma_j^2) \\ &\leq \frac{4\gamma_1^2 \exp(8\alpha l \gamma_1^2) 2^{4\alpha\gamma_1}}{(T+1)^{4\alpha\gamma_1}} \sum_{t=1}^T \frac{1}{j^{2-4\alpha\gamma_1}} \\ &\leq \frac{4\gamma_1^2 \exp(8\alpha l \gamma_1^2) 2^{4\alpha\gamma_1}}{(4\alpha\gamma_1-1)(T+1)} \end{aligned}$$

bringing all the pieces back together and substituting the corresponding bounds give the result under the conditional probability $\mathbb{P}_{\mathbf{n}}$. Taking the expectation over the distribution of the datas gives the result in terms of $\mathbb{E}[\sigma_n^2(\theta_n^*)]$. Since $\mathbb{E}_{\mathbf{n}}[g(\theta_{\mathbf{n}}^*)] = 0$, we have $\mathbb{E}[\sigma_n^2(\theta_n^*)] = \mathbb{E}[\|g(\theta_{\mathbf{n}}^*)\|^2] = \sigma^2(g(\theta_n^*))$ and we get the final result.

□

4 Additional Experimental Results

In the main text, we provided results in terms of the number of iterations. For completeness, Figure 1 shows the corresponding results with respect to the computational time. The same conclusions hold regarding the relative performance of the two approaches.

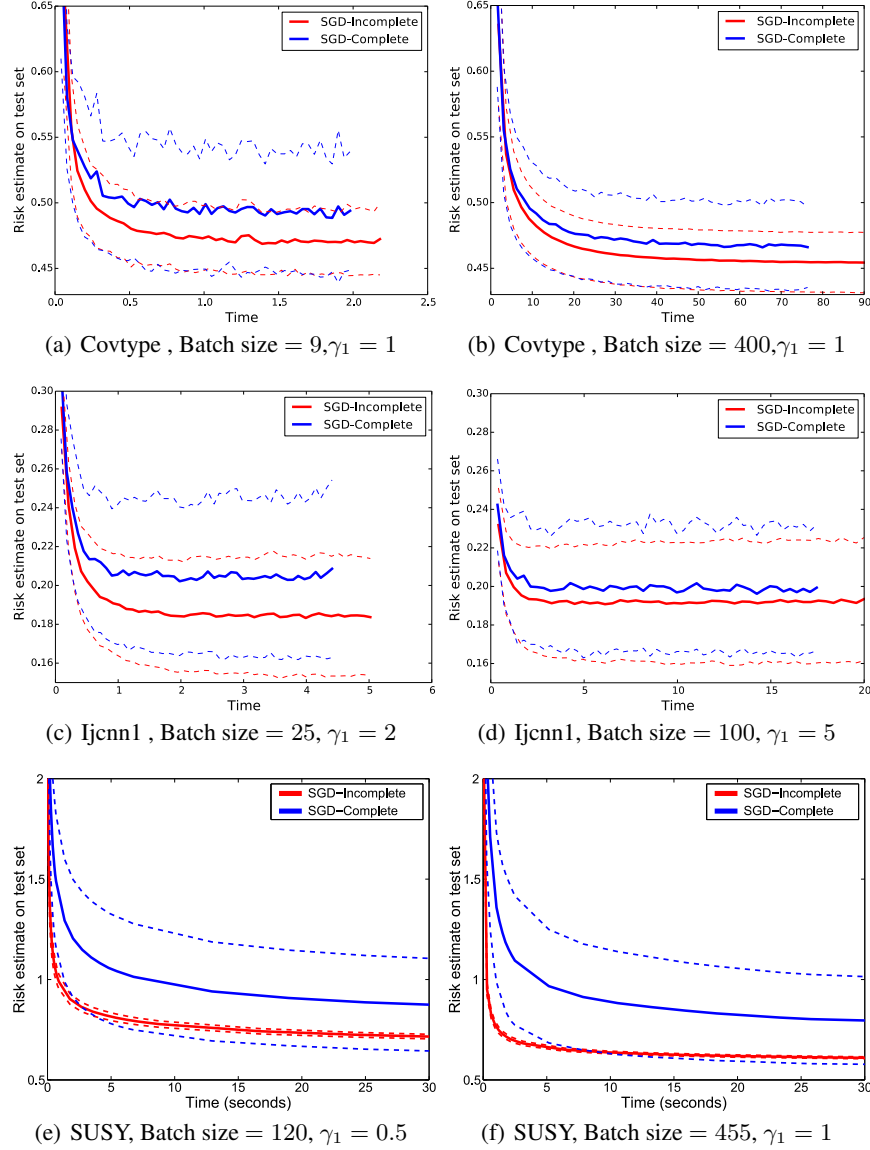


Figure 1: Average over 50 runs of risk estimate in terms of computational time in seconds (solid lines) +/- their standard deviation (dashed lines)

References

- [1] F. R. Bach and E. Moulines. Non-Asymptotic Analysis of Stochastic Approximation Algorithms for Machine Learning. In *NIPS*, 2011.
- [2] S. Cléménçon, G. Lugosi, and N. Vayatis. Ranking and empirical risk minimization of U-statistics. *Ann. Statist.*, 36, 2008.
- [3] S. Cléménçon, P. Bertail, and E. Chautru. Scaling up M-estimation via sampling designs: The Horvitz-Thompson stochastic gradient descent. In *IEEE Big Data*, 2014.
- [4] B. Delyon. Stochastic Approximation with Decreasing Gain: Convergence and Asymptotic Theory, 2000.
- [5] G. Fort. Central limit theorems for stochastic approximation with controlled Markov Chain. *EsaimPS*, 2014.
- [6] Y. Nesterov. *Introductory lectures on convex optimization*, volume 87. Springer, 2004.
- [7] Y. Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*. Applied Optimization. Springer, 2004.
- [8] M. Pelletier. Weak convergence rates for stochastic approximation with application to multiple targets and simulated annealing. *Ann. Appl.Prob.*, 1998.