A Proofs

Our main results utilize an elementary fact about smooth functions with Lipschitz continuous gradient, called the co-coercivity of the gradient. We state the lemma and recall its proof for completeness.

A.1 The Co-coercivity Lemma

Lemma A.1 (Co-coercivity) *For a smooth function f whose gradient has Lipschitz constant L,*

$$\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|_2^2 \leq L \left\langle \boldsymbol{x} - \boldsymbol{y}, \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}) \right\rangle.$$

Proof. Since ∇f has Lipschitz constant L, if $\boldsymbol{x}_\star$ is the minimizer of f, then

$$\frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{x}_\star)\|_2^2 = \frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{x}_\star)\|_2^2 + \langle \boldsymbol{x} - \boldsymbol{x}_\star, \nabla f(\boldsymbol{x}_\star)\rangle \leq f(\boldsymbol{x}) - f(\boldsymbol{x}_\star); \tag{A.1}$$

see, for instance, [[13], page 26]. Now define the convex functions

$$G(\boldsymbol{z}) = f(\boldsymbol{z}) - \langle \nabla f(\boldsymbol{x}), \boldsymbol{z}\rangle, \quad \text{and} \quad H(\boldsymbol{z}) = f(\boldsymbol{z}) - \langle \nabla f(\boldsymbol{y}), \boldsymbol{z}\rangle,$$

and observe that both have Lipschitz constants L and minimizers $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. Applying (A.1) to these functions therefore gives that

$$G(\boldsymbol{x}) \leq G(\boldsymbol{y}) - \frac{1}{2L}\|\nabla G(\boldsymbol{y})\|_2^2, \quad \text{and} \quad H(\boldsymbol{y}) \leq H(\boldsymbol{x}) - \frac{1}{2L}\|\nabla H(\boldsymbol{y})\|_2^2.$$

By their definitions, this implies that

$$\begin{aligned} f(\boldsymbol{x}) - \langle \nabla f(\boldsymbol{x}), \boldsymbol{x}\rangle &\leq f(\boldsymbol{y}) - \langle \nabla f(\boldsymbol{x}), \boldsymbol{y}\rangle - \frac{1}{2L}\|\nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x})\|_2^2 \\ f(\boldsymbol{y}) - \langle \nabla f(\boldsymbol{y}), \boldsymbol{y}\rangle &\leq f(\boldsymbol{x}) - \langle \nabla f(\boldsymbol{y}), \boldsymbol{x}\rangle - \frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|_2^2. \end{aligned}$$

Adding these two inequalities and canceling terms yields the desired result.

□

A.2 Proof of Theorem 2.1

With the notation of Theorem 2.1, and where i is the random index chosen at iteration k, we have

$$\begin{aligned} \|\boldsymbol{x}_{k+1} - \boldsymbol{x}_\star\|_2^2 &= \|\boldsymbol{x}_k - \boldsymbol{x}_\star - \gamma \nabla f_i(\boldsymbol{x}_k)\|_2^2 \\ &= \|(\boldsymbol{x}_k - \boldsymbol{x}_\star) - \gamma(\nabla f_i(\boldsymbol{x}_k) - \nabla f_i(\boldsymbol{x}_\star)) - \gamma \nabla f_i(\boldsymbol{x}_\star)\|_2^2 \\ &= \|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 - 2\gamma \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla f_i(\boldsymbol{x}_k)\right\rangle + \\ &\qquad \gamma^2 \|\nabla f_i(\boldsymbol{x}_k) - \nabla f_i(\boldsymbol{x}_\star) + \nabla f_i(\boldsymbol{x}_\star)\|_2^2 \\ &\leq \|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 - 2\gamma \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla f_i(\boldsymbol{x}_k)\right\rangle + \\ &\qquad 2\gamma^2 \|\nabla f_i(\boldsymbol{x}_k) - \nabla f_i(\boldsymbol{x}_\star)\|_2^2 + 2\gamma^2 \|\nabla f_i(\boldsymbol{x}_\star)\|_2^2 \\ &\leq \|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 - 2\gamma \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla f_i(\boldsymbol{x}_k)\right\rangle \\ &\qquad + 2\gamma^2 L_i \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla f_i(\boldsymbol{x}_k) - \nabla f_i(\boldsymbol{x}_\star)\right\rangle + 2\gamma^2 \|\nabla f_i(\boldsymbol{x}_\star)\|_2^2, \end{aligned}$$

where we have employed Jensen's inequality in the first inequality and the co-coercivity Lemma A.1 in the final line. We next take an expectation with respect to the choice of i. By assumption, $i \sim \mathcal{D}$ such that $F(\boldsymbol{x}) = \mathbb{E} f_i(\boldsymbol{x})$ and $\sigma^2 = \mathbb{E}\|\nabla f_i(\boldsymbol{x}_\star)\|^2$. Then $\mathbb{E}\nabla f_i(\boldsymbol{x}) = \nabla \boldsymbol{F}(\boldsymbol{x})$, and we obtain:

$$\begin{aligned} \mathbb{E}\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_\star\|_2^2 &\leq \|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 - 2\gamma \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla \boldsymbol{F}(\boldsymbol{x}_k)\right\rangle \\ &\qquad + 2\gamma^2 \mathbb{E}\left[L_i \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla f_i(\boldsymbol{x}_k) - \nabla f_i(\boldsymbol{x}_\star)\right\rangle\right] + 2\gamma^2 \mathbb{E}\|\nabla f_i(\boldsymbol{x}_\star)\|_2^2 \\ &\leq \|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 - 2\gamma \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla \boldsymbol{F}(\boldsymbol{x}_k)\right\rangle \\ &\qquad + 2\gamma^2 \sup_i L_i \mathbb{E}\left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla f_i(\boldsymbol{x}_k) - \nabla f_i(\boldsymbol{x}_\star)\right\rangle + 2\gamma^2 \mathbb{E}\|\nabla f_i(\boldsymbol{x}_\star)\|_2^2 \\ &= \|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 - 2\gamma \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla \boldsymbol{F}(\boldsymbol{x}_k)\right\rangle \\ &\qquad + 2\gamma^2 \sup L \left\langle \boldsymbol{x}_k - \boldsymbol{x}_\star, \nabla \boldsymbol{F}(\boldsymbol{x}_k) - \nabla \boldsymbol{F}(\boldsymbol{x}_\star)\right\rangle + 2\gamma^2 \sigma^2 \end{aligned}$$

We now utilize the strong convexity of $F(\boldsymbol{x})$ and obtain that

$$\begin{aligned}&\leq \|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 - 2\gamma\mu(1-\gamma\sup L)\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 + 2\gamma^2\sigma^2\\ &= (1 - 2\gamma\mu(1-\gamma\sup L))\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 + 2\gamma^2\sigma^2\end{aligned}$$

when $\gamma\mu \leq 1$. Recursively applying this bound over the first k iterations yields the desired result,

$$\begin{aligned}\mathbb{E}\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 &\leq \Big(1 - 2\gamma\mu(1-\gamma\sup L)\Big)\Big)^k \|\boldsymbol{x}_0 - \boldsymbol{x}_\star\|_2^2 + 2\sum_{j=0}^{k-1}\Big(1 - 2\gamma\mu(1-\gamma\sup L)\Big)\Big)^j \gamma^2\sigma^2\\ &\leq \Big(1 - 2\gamma\mu(1-\gamma\sup L)\Big)\Big)^k \|\boldsymbol{x}_0 - \boldsymbol{x}_\star\|_2^2 + \frac{\gamma\sigma^2}{\mu\big(1-\gamma\sup L\big)}.\end{aligned}$$

We next turn to the second part of the theorem, where we optimize the step size γ for a fixed tolerance ε. Recall the main recursive step in the previous proof,

$$\mathbb{E}\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_\star\|_2^2 \leq (1 - 2\mu\gamma(1-\gamma\sup L))\,\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2 + 2\gamma^2\sigma^2, \tag{A.2}$$

which is valid as long as $\mu\gamma \leq 1$. The minimal value of the quadratic

$$F_\xi(\gamma) = (1 - 2\gamma\mu(1-\gamma\sup L))\,\xi + 2\sigma^2\gamma^2$$

is achieved at

$$\gamma_\xi^* = \frac{\mu\xi}{2\xi\mu\sup L + 2\sigma^2}, \tag{A.3}$$

and

$$F_\xi(\gamma_\xi^*) = \big(1 - \frac{\mu^2\xi}{2\mu\sup L\xi + 2\sigma^2}\big)\xi \tag{A.4}$$

Note that because $\sup L/\mu \geq 1$, it follows that $\mu\gamma_\xi^* \leq 1/2$. Thus if we choose step-size $\gamma^* = \gamma_\varepsilon^*$,

$$\begin{aligned}\mathbb{E}\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_\star\|_2^2 &\leq F_{\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2}(\gamma^*) &\text{(A.5)}\\ &= \Big(F_{\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2}(\gamma^*) - F_\varepsilon(\gamma^*)\Big) + F_\varepsilon(\gamma^*) &\text{(A.6)}\\ &\leq \Big(1 - \frac{\mu^2\varepsilon}{2\mu\varepsilon\sup L + 2\sigma^2}\Big)\|\boldsymbol{x}_k - \boldsymbol{x}_\star\|_2^2. &\text{(A.7)}\\ & &\text{(A.8)}\end{aligned}$$

Iterating the expectation,

$$\mathbb{E}\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_\star\|_2^2 \leq \Big(1 - \frac{\mu^2\varepsilon}{2\mu\varepsilon\sup L + 2\sigma^2}\Big)^k \varepsilon_0. \tag{A.9}$$

It follows that if $\varepsilon \leq \mathbb{E}\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_\star\|_2^2$, then

$$\begin{aligned}\log(\varepsilon/\varepsilon_0) &\leq k\log\Big(1 - \frac{\mu^2\varepsilon}{2\mu\varepsilon\sup L + 2\sigma^2}\Big) &\text{(A.10)}\\ &\leq -k\big(\frac{\mu^2\varepsilon}{2\mu\sup L\varepsilon + 2\sigma^2}\big) &\text{(A.11)}\end{aligned}$$

or, equivalently

$$\begin{aligned}k &\leq \log(\varepsilon_0/\varepsilon)\big(\frac{2\mu\sup L\varepsilon + 2\sigma^2}{\mu^2\varepsilon}\big) &\text{(A.12)}\\ &= \log(\varepsilon_0/\varepsilon)\Big(\frac{2\sup L}{\mu} + \frac{2\sigma^2}{\mu^2\varepsilon}\Big). &\text{(A.13)}\end{aligned}$$

□