# Supplementary Material: Bayesian Inference for Structured Spike and Slab Priors

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The purpose of this supplementary document is to provide further details for the paper "Bayesian Inference for Structured Spike and Slab Priors".

### Model specification

The model is a standard linear model of the form

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e} \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}^N, \boldsymbol{x} \in \mathbb{R}^D, \boldsymbol{A} \in \mathbb{R}^{N \times D}$ and $\boldsymbol{e}\mathbb{R}^N$. The noise in the system is assumed to be i.i.d. Gaussian distributed

$$p(\boldsymbol{y}|\boldsymbol{x}) = \mathcal{N}\left(\boldsymbol{y}|\boldsymbol{A}\boldsymbol{x}, \sigma_0^2\boldsymbol{I}\right) \tag{2}$$

and we impose a spike and slab model on the prior

$$p(x_i|z_i) = (1 - z_i)\,\delta\left(x_i\right) + z_i\mathcal{N}\left(x_i\middle|0, \tau\right) \tag{3}$$

The support variables $\boldsymbol{z} = \{z_1, z_2, ..., z_D\}$ is assumed to be Bernoulli distribution

$$p(z_i|\gamma_i) = \text{Ber}\left(z_i\middle|\phi\left(\gamma_i\right)\right) \tag{4}$$

where $\phi : \mathbb{R} \rightarrow (0, 1)$ is the standard normal CDF function. Finally, we impose a multivariate Gaussian density on $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, ..., \gamma_D\}$

$$p(\boldsymbol{\gamma}) = \mathcal{N}\left(\boldsymbol{\gamma}\middle|\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\right) \tag{5}$$

The joint posterior over $\boldsymbol{x}, \boldsymbol{z}$ and $\boldsymbol{\gamma}$

$$p\left(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\gamma}\middle|\boldsymbol{y}\right) = \frac{1}{Z}\mathcal{N}\left(\boldsymbol{y}|\boldsymbol{A}\boldsymbol{x}, \sigma_0^2\boldsymbol{I}\right)\prod_{i=1}^{D}\left[(1 - z_i)\,\delta\left(x_i\right) + z_i\mathcal{N}\left(x_i\middle|0, \tau\right)\right]\prod_{i=1}^{D}\text{Ber}\left(z_i\middle|\phi\left(\gamma_i\right)\right)\mathcal{N}\left(\boldsymbol{\gamma}\middle|\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\right) \tag{6}$$

### Variational distribution for Expectation propagation

The joint variational distribution is of the form

$$\begin{aligned}
Q\left(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\gamma}\right) &= \mathcal{N}\left(\boldsymbol{x}\middle|\tilde{\boldsymbol{m}}, \tilde{\boldsymbol{V}}\right)\prod_{i=1}^{D}\text{Ber}\left(z_i\middle|\phi\left(\hat{\gamma}_i\right)\right)\mathcal{N}\left(\boldsymbol{\gamma}\middle|\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}\right) \\
&= \tilde{f}_1\left(\boldsymbol{x}\right)\tilde{f}_2\left(\boldsymbol{z}\right)\tilde{f}_3\left(\boldsymbol{z}, \boldsymbol{\gamma}\right)\tilde{f}_4\left(\boldsymbol{\gamma}\right) \\
&= \underbrace{\mathcal{N}\left(\boldsymbol{x}\middle|\tilde{\boldsymbol{m}}_1, \tilde{\boldsymbol{V}}_1\right)}_{\tilde{f}_1}\underbrace{\mathcal{N}\left(\boldsymbol{x}\middle|\tilde{\boldsymbol{m}}_2, \tilde{\boldsymbol{V}}_2\right)\prod_{i=1}^{D}\text{Ber}\left(z_i\middle|\phi\left(\tilde{\gamma}_{2,i}\right)\right)}_{\tilde{f}_2}\underbrace{\prod_{i=1}^{D}\text{Ber}\left(z_i\middle|\phi\left(\tilde{\gamma}_{3,i}\right)\right)\mathcal{N}\left(\boldsymbol{\gamma}\middle|\tilde{\boldsymbol{\mu}}_3, \tilde{\boldsymbol{\Sigma}}_3\right)}_{\tilde{f}_3}\underbrace{\mathcal{N}\left(\boldsymbol{\gamma}\middle|\tilde{\boldsymbol{\mu}}_4, \tilde{\boldsymbol{\Sigma}}_4\right)}_{\tilde{f}_4}
\end{aligned} \tag{7}$$

where $\tilde{\boldsymbol{V}}_2$ is diagonal with elements $\{\tilde{v}_{2,j}\}$ and similar for $\tilde{\boldsymbol{\Sigma}}_3$. We immediately make the following identifications. The first approximation term $\tilde{f}_1$ corresponds to the likelihood term and the fourth term $\tilde{f}_4$ corresponds to the prior on $\boldsymbol{\gamma}$. Thus, we only have to approximate the parameters in the second and third term, i.e. $\tilde{f}_2$ and $\tilde{f}_3$ .

**Computing joint approximation Q from $\tilde{f}_a$**

Given $\tilde{f}_i$ for $i = 1, 2, 3, 4$, we get:

$$\tilde{\boldsymbol{V}} = \left(\tilde{\boldsymbol{V}}_1^{-1} + \tilde{\boldsymbol{V}}_2^{-1}\right)^{-1} \tag{8}$$

$$\tilde{\boldsymbol{m}} = \left(\tilde{\boldsymbol{V}}_1^{-1}\tilde{\boldsymbol{m}}_1 + \tilde{\boldsymbol{V}}_2^{-1}\tilde{\boldsymbol{m}}_2\right) \tag{9}$$

$$\tilde{\boldsymbol{\Sigma}} = \left(\tilde{\boldsymbol{\Sigma}^{-1}}_3 + \tilde{\boldsymbol{\Sigma}}_4^{-1}\right)^{-1} \tag{10}$$

$$\tilde{\boldsymbol{\mu}} = \tilde{\boldsymbol{\Sigma}}\left(\tilde{\boldsymbol{\Sigma}}_3^{-1}\tilde{\boldsymbol{\mu}}_3 + \tilde{\boldsymbol{\Sigma}^{-1}}_4\tilde{\boldsymbol{\mu}}_4\right) \tag{11}$$

$$\tilde{\gamma}_j = t\left(\tilde{\gamma}_{2,j}, \tilde{\gamma}_{3,j}\right), \quad \forall j \in \{1, .., D\} \tag{12}$$

We have defined the following auxilary functions

$$d\left(x, y\right) = \phi^{-1}\left[\left(\frac{\left(1-\phi(x)\right)\phi(y)}{\left(1-\phi(y)\right)\phi(x)} + 1\right)^{-1}\right], \quad t\left(x, y\right) \quad = \phi^{-1}\left[\left(\frac{\left(1-\phi(x)\right)\left(1-\phi(y)\right)}{\phi(x)\phi(y)} + 1\right)^{-1}\right]$$

where $\phi^{-1}(x)$ is the probit function. The function $t(\cdot,\cdot)$ amounts to computing the product of two Bernoulli densities parametrized using $\phi\left(\cdot\right)$ and $d(\cdot,\cdot)$ is the corresponding function for quotients of Bernoulli densities.

**Computing the cavity distributions for $\tilde{f}_2$**

$$Q^{\backslash 2,j}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\gamma}) = \frac{Q(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\gamma})}{f_{2,j}(x_j, z_j)} = \frac{\mathcal{N}\left(\boldsymbol{x}\middle|\tilde{\boldsymbol{m}}, \tilde{\boldsymbol{V}}\right)\prod_{i=1}^{D}\mathrm{Ber}\left(z_i\middle|\phi\left(\tilde{\gamma}_i\right)\right)\mathcal{N}\left(\tilde{\boldsymbol{\gamma}}\middle|\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}\right)}{\mathrm{Ber}\left(z_j\middle|\phi\left(\tilde{\gamma}_{2,j}\right)\right)\mathcal{N}\left(x_j\middle|\tilde{m}_{2,j}, \tilde{v}_{2,j}\right)} \tag{13}$$

The parameters for the $j$'th marginal cavity distribution then becomes:

$$v_j^{\backslash 2,j} = \left(\left(\tilde{V}_{jj}\right)^{-1} - \tilde{v}_{2,j}^{-1}\right)^{-1} \tag{14}$$

$$m_j^{\backslash 2,j} = v_j^{\backslash 2,j}\left(\left(\tilde{V}_{jj}\right)^{-1}\tilde{m}_j - \tilde{v}_{2,j}^{-1}\tilde{m}_{2,j}\right), \tag{15}$$

$$\gamma^{\backslash 2,j} = d\left(\tilde{\gamma}_j, \tilde{\gamma}_{2,j}\right) \tag{16}$$

**Moment matching for the second term**

Computing the normalization for $f_{2,j}Q^{\backslash 2,j}$:

$$\begin{aligned} Z_{2,j} &= \sum_{\boldsymbol{z}}\int\int f_{2,j}(x_j, z_j)Q^{\backslash 2,j}\left(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\gamma}\right)\mathrm{d}\boldsymbol{\gamma}\mathrm{d}\boldsymbol{x} \\ &= \sum_{z_j}\int f_{2,j}(x_j, z_j)\sum_{\boldsymbol{z}_{\backslash j}}\int\int Q^{\backslash 2,j}\left(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\gamma}\right)\mathrm{d}\boldsymbol{\gamma}\mathrm{d}\boldsymbol{x}_{\backslash j}\mathrm{d}x_j \end{aligned} \tag{17}$$

Plugging in the densities

$$\begin{aligned} Z_{2,j} &= \sum_{z_j}\int\left[(1-z_j)\delta(x_j) + z_j\mathcal{N}\left(x_j\middle|0, \tau\right)\right]\sum_{\boldsymbol{z}_{\backslash j}}\int\int\mathcal{N}\left(\boldsymbol{x}\middle|\boldsymbol{m}^{\backslash 2,j}, \boldsymbol{V}^{\backslash 2,j}\right)\prod_{i=1}^{D}\mathrm{Ber}\left(z_i\middle|\phi\left(\gamma_i^{\backslash 2,j}\right)\right)\mathcal{N}\left(\boldsymbol{\gamma}\middle|\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}\right)\mathrm{d}\boldsymbol{\gamma}\mathrm{d}\boldsymbol{x}_{\backslash j} \\ &= \sum_{z_j}\int\left[(1-z_j)\delta(x_j) + z_j\mathcal{N}\left(x_j\middle|0, \tau\right)\right]\mathrm{Ber}\left(z_j\middle|\phi\left(\gamma_j^{\backslash 2,j}\right)\right)\mathcal{N}\left(x_j\middle|m_j^{\backslash 2,j}, v_j^{\backslash 2,j}\right)\mathrm{d}x_j \\ &= \left(1-\phi\left(\gamma_j^{\backslash 2,j}\right)\right)\mathcal{N}\left(0\middle|m_j^{\backslash 2,j}, v_{jj}^{\backslash 2,j}\right) + \phi\left(\gamma_j^{\backslash 2,j}\right)\mathcal{N}\left(x_j\middle|m_j^{\backslash 2,j}, \tau + v_j^{\backslash 2,j}\right) \end{aligned} \tag{18}$$

For the moment matching, we need the following moments:

$$\mathbb{E}_{f_{2,j}Q^{\setminus 2,j}}\left[\boldsymbol{x}\right], \quad \mathbb{E}_{f_{2,j}Q^{\setminus 2,j}}\left[\boldsymbol{x}\boldsymbol{x}^T\right], \quad \& \quad \mathbb{E}_{f_{2,j}Q^{\setminus 2,j}}\left[\boldsymbol{z}\right] \tag{19}$$

The moment matching results in the following update equations:

$$\begin{aligned}
v_{jj}^{\text{new}} &= \frac{a_j}{a_j+b_j}\left[\frac{\left(m_j^{\setminus 2,j}\tau\right)^2}{\left(v_j^{\setminus 2,j}+\tau\right)^2} + \frac{v_j^{\setminus 2,j}\tau}{v_j^{\setminus 2,j}+\tau}\right] - \left[\frac{a_j}{a_j+b_j}\frac{m_j^{\setminus 2,j}\tau}{v_j^{\setminus 2,j}+\tau}\right]^2 \\
\tilde{v}_{2,j}^{\text{new}} &= \left[\left(v_{jj}^{\text{new}}\right)^{-1} - \left(v_j^{\setminus 2,j}\right)^{-1}\right]^{-1} && (20) \\
m_j^{\text{new}} &= \frac{a_j}{a_j+b_j}\frac{m_j^{\setminus 2,j}\tau}{v_j^{\setminus 2,j}+\tau} && (21) \\
\tilde{m}_{2,j}^{\text{new}} &= v_{2,j}^{\text{new}}\left(\left(v_j^{\text{new}}\right)^{-1}m_j^{\text{new}} - \left(v_j^{\setminus 2,j}\right)^{-1}m_j^{\setminus 2,j}\right) && (22) \\
\tilde{\gamma}_j^{\text{new}} &= \phi^{-1}\left(\frac{a_j}{a_j+b_j}\right) && (23) \\
\tilde{\gamma}_j^{\text{new}} &= d\left(\tilde{\gamma}_j^{\text{new}}, \gamma_j^{\setminus 2,j}\right) && (24)
\end{aligned}$$

where

$$\begin{aligned}
a_j &= \phi\left(\gamma_j^{\setminus 2,j}\right)\mathcal{N}\left(0\middle|m_j^{\setminus 2,j}, \tau+v_j^{\setminus 2,j}\right) && (25) \\
b_j &= \left(1-\phi\left(\gamma_j^{\setminus 2,j}\right)\right)\mathcal{N}\left(0\middle|m_j^{\setminus 2,j}, v_{jj}^{\setminus 2,j}\right) && (26)
\end{aligned}$$

**Computing the cavity distributions for the third term**

$$Q^{\setminus 3,j}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\gamma}) = \frac{Q(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\gamma})}{f_{3,j}(z_j,\gamma_j)} = \frac{\mathcal{N}\left(\boldsymbol{x}\middle|\tilde{\boldsymbol{m}},\tilde{\boldsymbol{V}}\right)\prod_{i=1}^{D}\text{Ber}\left(z_i\middle|\phi\left(\tilde{\gamma}_i\right)\right)\mathcal{N}\left(\boldsymbol{\gamma}\middle|\tilde{\boldsymbol{\mu}},\tilde{\boldsymbol{\Sigma}}\right)}{\text{Ber}\left(z_j\middle|\phi\left(\tilde{\gamma}_{3,j}\right)\right)\mathcal{N}\left(\gamma_j\middle|\tilde{\mu}_{3,j},\tilde{\Sigma}_{3,j}\right)} \tag{27}$$

The parameters for the $j$'th marginal cavity distribution then becomes

$$\begin{aligned}
\Sigma_j^{\setminus 3,j} &= \left(\left(\tilde{V}_{jj}\right)^{-1} - \left(\tilde{\Sigma}_{3,j}\right)^{-1}\right)^{-1} && (28) \\
\mu_j^{\setminus 3,j} &= \Sigma_j^{\setminus 3,j}\left(\left(\tilde{V}_{jj}\right)^{-1}\tilde{m}_j - \left(\tilde{\Sigma}_{3,j}\right)^{-1}\tilde{\mu}_{3,j}\right), && (29) \\
\gamma_j^{\setminus 3,j} &= d\left(\tilde{\gamma}_j, \tilde{\gamma}_{3,j}\right) && (30)
\end{aligned}$$

**Moment matching for the third term**

We need to following moments

$$\mathbb{E}_{f_{3,j}Q^{\setminus 3,j}}\left[\boldsymbol{\gamma}\right], \quad \mathbb{E}_{f_{3,j}Q^{\setminus 3,j}}\left[\boldsymbol{\gamma}\boldsymbol{\gamma}^T\right], \quad \& \quad \phi\left(\boldsymbol{\gamma}\right)^{\text{new}} = \mathbb{E}_{f_{3,j}Q^{\setminus 3,j}}\left[\boldsymbol{z}\right] \tag{31}$$

Computing the normalization for $f_{3,j}Q^{\setminus 3,j}$

$$\begin{aligned}
Z_{3,j} &= \sum_{\boldsymbol{z}}\int\int f_{3,j}(z_j,\gamma_j)Q^{\setminus 3,j}\left(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\gamma}\right)\mathrm{d}\boldsymbol{\gamma}\mathrm{d}\boldsymbol{x} \\
&= \sum_{z_j}\int f_{3,j}(z_j,\gamma_j)\sum_{\boldsymbol{z}_{\setminus j}}\int\int Q^{\setminus 3,j}\left(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\gamma}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\gamma}_{\setminus j}\mathrm{d}\gamma_j \\
&= \sum_{z_j}\int f_{3,j}(z_j,\gamma_j)Q^{\setminus 3,j}\left(z_j,\gamma_j\right)\mathrm{d}\gamma_j && (32)
\end{aligned}$$

where $\boldsymbol{x}, \boldsymbol{z}_{\setminus j}$ and $\boldsymbol{\gamma}_{\setminus j}$ are marginalized out. Plugging in the densities

$$Z_{3,j} = \sum_{z_j} \int Ber\left(z_j|\phi\left(\gamma_j\right)\right) \mathrm{Ber}\left(z_j|\phi\left(\gamma_j^{\setminus 3,j}\right)\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \tag{33}$$

Carrying out the sum

$$Z_{3,j} = \int \left[\left(1-\phi\left(\gamma_j\right)\right)\left(1-\phi\left(\gamma_j^{\setminus 3,j}\right)\right) + \phi\left(\gamma_j\right)\phi\left(\gamma_j^{\setminus 3,j}\right)\right] \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \tag{34}$$

Applying linearity of integrals

$$\begin{aligned} Z_{3,j} = &\left(1-\phi\left(\gamma_j^{\setminus 3,j}\right)\right) \int \left(1-\phi\left(\gamma_j\right)\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \\ &+ \phi\left(\gamma_j^{\setminus 3,j}\right) \int \phi\left(\gamma_j\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \end{aligned} \tag{35}$$

The integral evaluate to

$$\int \phi\left(\gamma_j\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j = \phi\left(\frac{\mu_j^{\setminus 3,j}}{\sqrt{1+\boldsymbol{\Sigma}_j^{\setminus 3,j}}}\right) = \phi(z) \equiv C_{3,j} \tag{36}$$

where we have defined $z$ as

$$z = \frac{\mu_j^{\setminus 3,j}}{\sqrt{1+\boldsymbol{\Sigma}_j^{\setminus 3,j}}} \tag{37}$$

Therefore, the normalization becomes

$$Z_{3,j} = \left(1-\phi\left(\gamma_j^{\setminus 3,j}\right)\right)\left(1-C_{3,j}\right) + \phi\left(\gamma_j^{\setminus 3,j}\right) C_{3,j} \tag{38}$$

Similarly, we compute the first moment of $\boldsymbol{z}$

$$\begin{aligned} \phi\left(\gamma_j\right)^{\text{new}} &= \frac{1}{Z_{3,j}} \sum_{z_j} \int z_j \cdot f_{3,j}(z_j,\gamma_j) Q^{\setminus 3,j}\left(z_j,\gamma_j\right) \mathrm{d}\gamma_j \\ &= \frac{1}{Z_{3,j}} \sum_{z_j} \int z_j \mathrm{Ber}\left(z_j|\phi\left(\gamma_j\right)\right) \mathrm{Ber}\left(z_j|\phi\left(\gamma_j^{\setminus 3,j}\right)\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \\ &= \frac{1}{Z_{3,j}} \phi\left(\gamma_j^{\setminus 3,j}\right) \int \phi\left(\gamma_j\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \\ &= \frac{1}{Z_{3,j}} \phi\left(\gamma_j^{\setminus 3,j}\right) C_{3,j} \end{aligned} \tag{39}$$

where the result in eq. (36) is used. Therefore

$$\gamma_j^{\text{new}} = \phi^{-1}\left(\frac{1}{Z_{3,j}} \phi\left(\gamma_j^{\setminus 3,j}\right) C_{3,j}\right) \tag{40}$$

Hence, the update becomes

$$\tilde{\gamma}_{3,j} = d\left(\gamma_j^{\text{new}}, \gamma_j^{\setminus 3,j}\right) \tag{41}$$

Similarly, the first moment w.r.t. $\gamma_j$ evaluates to

$$\begin{aligned} \mu_j^{\text{new}} &= \frac{1}{Z_{3,j}} \sum_{z_j} \int \gamma_j \mathrm{Ber}\left(z_j|\phi\left(\gamma_j\right)\right) \mathrm{Ber}\left(z_j|\phi\left(\gamma_j^{\setminus 3,j}\right)\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \\ &= \frac{1}{Z_{3,j}} \int \gamma_j \left[\left(1-\phi\left(\gamma_j\right)\right)\left(1-\phi\left(\gamma_j^{\setminus 3,j}\right)\right) + \phi\left(\gamma_j\right)\phi\left(\gamma_j^{\setminus 3,j}\right)\right] \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \\ &= \frac{1}{Z_{3,j}} \left(1-\phi\left(\gamma_j^{\setminus 3,j}\right)\right) \int \gamma_j \left(1-\phi\left(\gamma_j\right)\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \\ &\quad + \frac{1}{Z_{3,j}} \phi\left(\gamma_j^{\setminus 3,j}\right) \int \gamma_j \phi\left(\gamma_j\right) \mathcal{N}\left(\gamma_j|\mu_j^{\setminus 3,j}, \boldsymbol{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \end{aligned} \tag{42}$$

Using the the results from ch. 3.9 in the Gaussian process book (www.gpml.org), we have the following result:

$$\int \gamma_j \phi(\gamma_j) \mathcal{N}\left(\gamma_j \middle| \mu_j^{\setminus 3,j}, \mathbf{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j = C_{3,j} \left( \mu_j^{\setminus 3,j} + \frac{\mathbf{\Sigma}_j^{\setminus 3,j} \mathcal{N}\left(z \middle| 0, 1\right)}{C_{3,j} \sqrt{1 + \mathbf{\Sigma}_j^{\setminus 3,j}}} \right) \tag{43}$$

Note, that the scaling $C_{3,j}$ is necessary since the integral is not normalized. Now define

$$W_{3,j} = C_{3,j} \mu_j^{\setminus 3,j} + \frac{\mathbf{\Sigma}_j^{\setminus 3,j} \mathcal{N}\left(z \middle| 0, 1\right)}{\sqrt{1 + \mathbf{\Sigma}_j^{\setminus 3,j}}} \tag{44}$$

and inserting this result yields the expression for the first moment

$$\mu_j^{\text{new}} = \frac{1}{Z_{3,j}} \left[ \left(1 - \phi\left(\gamma_j^{\setminus 3,j}\right)\right) \left[\mu_j^{\setminus 3,j} - W_{3,j}\right] + \phi\left(\gamma_j^{\setminus 3,j}\right) W_{3,j} \right] \tag{45}$$

And for the second moment, we get

$$\begin{aligned}
\mathbb{E}_{f_{3,j} Q^{\setminus 3,j}}\left[\gamma_j^2\right] &= \frac{1}{Z_{3,j}} \sum_{z_j} \int \gamma_j^2 \mathrm{Ber}\left(z_j \middle| \phi(\gamma_j)\right) \mathrm{Ber}\left(z_j \middle| \phi\left(\gamma_j^{\setminus 3,j}\right)\right) \mathcal{N}\left(\gamma_j \middle| \mu_j^{\setminus 3,j}, \mathbf{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \\
&= \frac{1}{Z_{3,j}} \left(1 - \phi\left(\gamma_j^{\setminus 3,j}\right)\right) \left[ \int \gamma_j^2 \mathcal{N}\left(\gamma_j \middle| \mu_j^{\setminus 3,j}, \mathbf{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j - \int \gamma_j^2 \phi(\gamma_j) \mathcal{N}\left(\gamma_j \middle| \mu_j^{\setminus 3,j}, \mathbf{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j \right] \\
&\quad + \frac{1}{Z_{3,j}} \phi\left(\gamma_j^{\setminus 3,j}\right) \int \gamma_j^2 \phi(\gamma_j) \mathcal{N}\left(\gamma_j \middle| \mu_j^{\setminus 3,j}, \mathbf{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j
\end{aligned} \tag{46}$$

Using the same result for the integral, we get the following result

$$\int \gamma_j^2 \phi(\gamma_j) \mathcal{N}\left(\gamma_j \middle| \mu_j^{\setminus 3,j}, \mathbf{\Sigma}_j^{\setminus 3,j}\right) \mathrm{d}\gamma_j = 2\mu_j^{\setminus 3,j} W_{3,j} + C_{3,j} \left[ \mathbf{\Sigma}_j^{\setminus 3,j} - \left(\mu_j^{\setminus 3,j}\right)^2 \right] - \frac{\left(\mathbf{\Sigma}_j^{\setminus 3,j}\right)^2 z \mathcal{N}\left(z \middle| 0, 1\right)}{\left(1 + \mathbf{\Sigma}_j^{\setminus 3,j}\right)} \equiv M_{3,j}$$

Inserting $M_{3,j}$:

$$\mathbb{E}_{f_{3,j} Q^{\setminus 3,j}}\left[\gamma_j^2\right] = \frac{1}{Z_{3,j}} \left[ \left(1 - \phi\left(\gamma_j^{\setminus 3,j}\right)\right) \left[ \left(\mu_j^{\setminus 3,j}\right)^2 + \mathbf{\Sigma}_j^{\setminus 3,j} - M_{3,j} \right] + \phi\left(\gamma_j^{\setminus 3,j}\right) M_{3,j} \right] \tag{47}$$

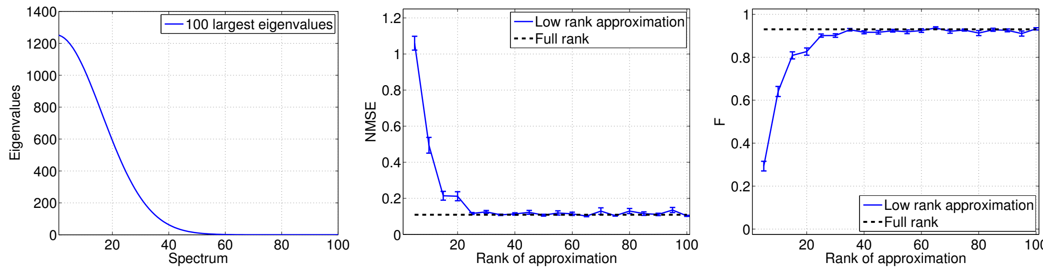
Finally, we obtain the variance as:

$$\Sigma_j^{\text{new}} = \mathbb{E}_{f_{3,j} Q^{\setminus 3,j}}\left[\gamma_j^2\right] - \mathbb{E}_{f_{3,j} Q^{\setminus 3,j}}\left[\gamma_j\right]^2 \tag{48}$$

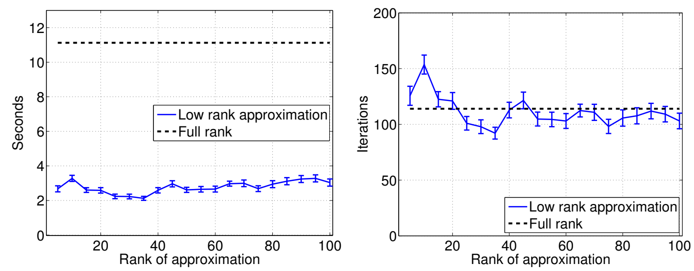
The update equations for the mean and variance are then given by:

$$\tilde{\Sigma}_{3,j}^{\text{new}} = \left[ \left(\Sigma_j^{\text{new}}\right)^{-1} - \left(\Sigma^{\setminus 3,j}\right)^{-1} \right]^{-1} \tag{49}$$

$$\tilde{\mu}_{3,j}^{\text{new}} = \tilde{\Sigma}_{3,j}^{\text{new}} \left( \left(\Sigma_j^{\text{new}}\right)^{-1} \mu_j^{\text{new}} - \left(\Sigma^{\setminus 3,j}\right)^{-1} \mu_j^{\setminus 3,j} \right) \tag{50}$$




(a) Spectrum (b) NMSE (c) F-measure




(d) Run time (e) Iterations

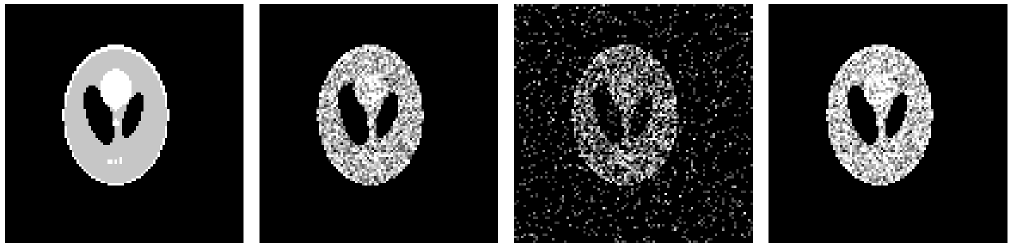
Figure 1: Illustration of the properties of the low rank approximation of $\mathbf{\Sigma}_0$. Data are generated the way as described for experiment 1 in the paper, except that $\boldsymbol{A} \in \mathbb{R}^{125\times 500}$ and the sparsity level is fixed at $K/D = 0.1$.

**Experiment: Effect of low rank approximation**

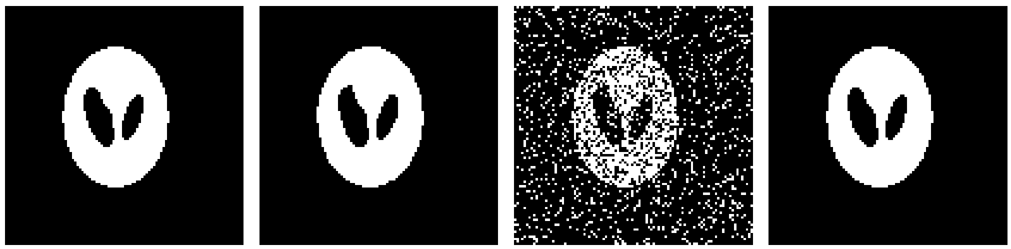
This experiment is designed to investigate the properties and implications of the low rank approximation of $\mathbf{\Sigma}_0$. We generate the problems in the same ways as in the first experiment (described in the paper), but now sweep over the rank $R$ of the approximation of $\mathbf{\Sigma}_0$ with fixed problem size, i.e. $N = 125$ and $D = 500$. The results are shown in figure 1. For the specific choice of covariance function, figure 1(b)-(c) shows that a 40-rank approximation does not introduce significant errors in terms of NMSE and F-measure. but the run time is reduced by a factor $\approx 3.5$. The reduction is expected to become even more significant as $D$ increases.

**Experiment: Shepp Logan Recovery**

We have also recreated the Shepp-Logan Phantom experiment from [1] with $D = 10^4$ unknowns, $K = 1723$ non-zero weights, $N = 2K$ observations and SNR $= 10dB$. That is, we generated a set of measurements using the model $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_0 + \boldsymbol{e}$, where the true signal $\boldsymbol{x}_0$ is the Shepp-Logan Phantom image (see figure 2a). For the EP method, we imposed a squared exponential covariance function with length-scale 8 for $\boldsymbol{\gamma}$ defined on the $100 \times 100$ image grid. We use three methods for reconstruction $\boldsymbol{x}_0$: Our proposed method, BG-AMP [2] and an oracle estimator, which computes a least squares estimate based on knowledge of the true support. We consider the Normalized Mean Square Error (NMSE) of the estimated coefficients $\hat{\boldsymbol{x}}$ as well as the F-measure of the estimated support $\hat{z}$. The reconstructions are shown in figure 2, where the first row shows the reconstructed coefficients and the second row shows the reconstructed support. Our proposed method yields $F_{sq} = 0.994$ and NMSE$_{sq}$ $= 0.336$ for this experiment, whereas BG-AMP yields $F = 0.624$ and NMSE $= 0.717$. For reference, the oracle estimator yields NMSE $= 0.326$.



(a) True coefficients $\boldsymbol{x}_0$ (b) $\hat{\boldsymbol{x}}_{\text{EP}}$ (c) $\hat{\boldsymbol{x}}_{\text{BG-AMP}}$ (d) $\hat{\boldsymbol{x}}_{\text{oracle}}$



(e) True support $\boldsymbol{z}_0$ (f) $\hat{\boldsymbol{z}}_{\text{EP}}$ (g) $\hat{\boldsymbol{z}}_{\text{BG-AMP}}$ (h) $\hat{\boldsymbol{z}}_{\text{oracle}}$

Figure 2: Recovery of the Shepp-Logan Phantom. The first row shows the reconstructed coefficients $\hat{x}$ and the second row shows the reconstructed support $\hat{z}$.

## References


[1] V. Cevher, M. F. Duarte, C. Hegde, and R. G. Baraniuk. Sparse signal recovery using markov random fields. In *NIPS*, Vancouver, B.C., Canada, 8–11 December 2008.

[2] P. Schniter and J. Vila. Expectation-maximization gaussian-mixture approximate message passing. *2012 46th Annual Conference on Information Sciences and Systems, CISS 2012*, pages –, 2012.