Random Conic Pursuit for Semidefinite Programming: Supplementary Materials

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Theorem 2. *Let $X^* \succ 0$ be a fixed positive definite matrix, and let $x_1, \ldots, x_n \in \mathbb{R}^d$ be drawn i.i.d. from $\mathcal{N}(0, \Sigma)$ with $\Sigma \succ X^*$. Then, for any $\delta > 0$, with probability at least $1 - \delta$,*

$$\min_{X \in \mathcal{F}_n^x} \|X - X^*\| \leq \frac{1 + \sqrt{2} \log \frac{1}{\delta}}{\sqrt{n}} \cdot \frac{2}{e} \sqrt{\left|\Sigma X^{*-1}\right|} \left\| \left(X^{*-1} - \Sigma^{-1}\right)^{-1} \right\|_2$$

Proof of Theorem 2. We wish to bound the tails of the random variable

$$\min_{\gamma_1 \cdots \gamma_n \geq 0} \left\| \sum_{i=1}^{n} \gamma_i x_i x_i - X^* \right\|. \tag{1}$$

We first simplify the problem by eliminating the minimization over γ. Define a function $\gamma(x; X^*) : \mathbb{R}^d \to \mathbb{R}_+$ that satisfies

$$\mathop{\mathbb{E}}_{x \sim \mathcal{N}(0, \Sigma)} \gamma(x) x x' = X^*. \tag{2}$$

The choice

$$\gamma(x) = \frac{\mathcal{N}(x|0, X^*)}{\mathcal{N}(x|0, \Sigma)} = |\Sigma|^{1/2} \left|X^*\right|^{-1/2} \exp\left(-\tfrac{x'(X^{*-1} - \Sigma^{-1})x}{2}\right) \tag{3}$$

works, since $\mathbb{E}_{x \sim \mathcal{N}(0,\Sigma)} \gamma(x) x x' = \mathbb{E}_{x \sim \mathcal{N}(0, X^*)} x x' = X^*$. Setting sub-optimally the coefficients γ_i to $\gamma(x_i)$ gives

$$\begin{aligned} \min_{\gamma \geq 0} \left\| \sum_{i=1}^{n} \gamma_i x_i x_i' - X^* \right\| &\leq \left\| \tfrac{1}{n} \sum_{i=1}^{n} \gamma(x_i) x_i x_i' - \mathbb{E}\, \gamma(x) x x' \right\| && (4) \\ &= \left\| \tfrac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\|. && (5) \end{aligned}$$

So it suffices to bound the tails of the deviation of an empirical average of i.i.d. random variables $z_i := \gamma(x_i) x_i x_i'$ from its expectation, $\mathbb{E}\, z = X^*$. We proceed using McDiarmid's inequality.

The scalar random variables $\|z_i\|$ are bounded because for all x, we have:

$$\begin{aligned} \|z\| = \|\gamma(x) x x'\| &= \|x x'\| |\Sigma|^{\frac{1}{2}} \left|X^*\right|^{-\frac{1}{2}} \exp\left(-\tfrac{x'(X^{*-1} - \Sigma^{-1})x}{2}\right) && (6) \\ &\leq |\Sigma|^{\frac{1}{2}} \left|X^*\right|^{-\frac{1}{2}} \|x\|_2^2 \exp\left(-\tfrac{\lambda_{min}(X^{*-1} - \Sigma^{-1}) \|x\|_2^2}{2}\right) && (7) \\ &\leq \frac{2 |\Sigma|^{\frac{1}{2}} \left|X^*\right|^{-\frac{1}{2}}}{e \lambda_{min}(X^{*-1} - \Sigma^{-1})} && (8) \\ &= \frac{2}{e} |\Sigma|^{\frac{1}{2}} \left|X^*\right|^{-\frac{1}{2}} \left\| \left(X^{*-1} - \Sigma^{-1}\right)^{-1} \right\|_2 && (9) \\ &=: \Delta. && (10) \end{aligned}$$

Equation (8) follows because the function $f(y) = ye^{-\alpha y}$ is bounded above by $\frac{1}{e\alpha}$.

The expectation of $\left\| \frac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\|$, whose tails we wish to bound, is the standard deviation of $\frac{1}{n} \sum_{i=1}^{n} z_i$, and can be bounded in the standard way in a Hilbert space:

$$\left(\mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\| \right)^2 \leq \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\|^2 = \frac{1}{n} \left(\mathbb{E} \, \|z\|^2 - \| \mathbb{E}\, z \|^2 \right), \tag{11}$$

which yields

$$\mathbb{E} \left\| \tfrac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\| \leq \frac{\Delta}{\sqrt{n}}. \tag{12}$$

Using Equations (5) and (12), and the fact that $\|z_i\| \leq \Delta$, McDiarmid's inequality gives

$$\begin{aligned}
& \Pr \left[\min_{\gamma \geq 0} \left\| \sum_{i=1}^{n} \gamma_i x_i x_i' - X^* \right\| > \frac{\Delta}{\sqrt{n}} + \epsilon \right] && (13) \\
& \leq \Pr \left[\left\| \tfrac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\| > \frac{\Delta}{\sqrt{n}} + \epsilon \right] && (14) \\
& \leq \Pr \left[\left\| \tfrac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\| > \mathbb{E} \left\| \tfrac{1}{n} \sum_{i=1}^{n} z_i - \mathbb{E}\, z \right\| + \epsilon \right] \\
& \leq \exp \left(- \frac{n\epsilon^2}{2\Delta^2} \right). && (15)
\end{aligned}$$

In other words, for any $\delta > 0$, with probability at least $1 - \delta$,

$$\min_{\gamma \geq 0} \left\| \sum_{i=1}^{n} \gamma_i x_i x_i' - X^* \right\| < \frac{\Delta}{\sqrt{n}} \left(1 + \sqrt{2} \log \tfrac{1}{\delta} \right). \tag{16}$$

□