This suplementary material provides the proofs for the NIPS'10 paper *A POMDP Extension with Belief-dependent Rewards.*

A Proof of Theorem 3.1

Proof. Let us assume that $V_{n-1}(b)$ is a convex function, then we can show that $V_n(b)$ is also convex as follows. First, the value function in Eq. 1 can be expressed by parts as,

$$V_n(b) = \max_a \left[V_n^a(b)\right],$$
$$V_n^a(b) = \sum_o V_n^{a,o}(b),$$
$$V_n^{a,o}(b) = \frac{\rho(b,a)}{|\Omega|} + \psi^{a,o}(b), \text{ and}$$
$$\psi^{a,o}(b) = \gamma Pr(o|a,b)V_{n-1}(b^{a,o}).$$

Also, using the belief update, we can express $b^{a,o} = \frac{P^{a,o}\cdot b}{\|P^{a,o}\cdot b\|_1}$ with $P^{a,o}_{s,s'} = O(s',a,o)T(s,a,s')$. Therefore,

$$\psi^{a,o}(b) = \gamma\|P^{a,o}\cdot b\|_1 V_{n-1}(\frac{P^{a,o}\cdot b}{\|P^{a,o}\cdot b\|_1}) = \gamma\kappa(P^{a,o}\cdot b)$$

with $\kappa(w) = \|w\|_1 V_{n-1}(\frac{w}{\|w\|_1})$.

Here, $\kappa(w)$ is a convex function as it uses the perspective and linear-fractional convexity preserving operations (see App. A.1 for a stand-alone proof). Then, $\psi^{a,o}$ is also convex since convexity is preserved under affine maps. Consequently, $V_n^{a,o}$, V_n^a and V_n are convex because ρ and $\psi^{a,o}$ are convex.[6] Considering this last result is the inductive step, then $V_n(b)$ is convex over b, because V_0 is convex by definition. □

A.1 Stand-alone Proof of κ Convexity

Lemma A.1. *Let $w \in \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ a convex function. If $\kappa(w) = \|w\|_1 f(\frac{w}{\|w\|_1})$, then $\kappa(w)$ is also a convex function.*

Proof. As stated before, one can use the perspective and linear-fractional convexity preserving operations to directly prove that κ is convex. However, this can also be proved by only using the convexity of f:

$$\begin{aligned}
&\kappa(\alpha x + (1-\alpha)y) \\
&= \|\alpha x + (1-\alpha)y\|_1 f(\frac{\alpha x + (1-\alpha)y}{\|\alpha x + (1-\alpha)y\|_1}) \\
&= \|\alpha x + (1-\alpha)y\|_1 f(\frac{\alpha\|x\|_1}{\|\alpha x + (1-\alpha)y\|_1}\cdot\frac{x}{\|x\|_1} + \frac{(1-\alpha)\|y\|_1}{\|\alpha x + (1-\alpha)y\|_1}\cdot\frac{y}{\|y\|_1}) \\
&\leq \|\alpha x + (1-\alpha)y\|_1 (\frac{\alpha\|x\|_1}{\|\alpha x + (1-\alpha)y\|_1} f(\frac{x}{\|x\|_1}) + \frac{(1-\alpha)\|y\|_1}{\|\alpha x + (1-\alpha)y\|_1} f(\frac{y}{\|y\|_1})) \\
&= \alpha\|x\|_1 f(\frac{x}{\|x\|_1}) + (1-\alpha)\|y\|_1 f(\frac{y}{\|y\|_1}) \\
&= \alpha\kappa(x) + (1-\alpha)\kappa(y).
\end{aligned}$$

□

[6] Convex functions are closed under the sum and the max operators.

B Proofs of Section 4

B.1 Proof of Lemma 4.1

Proof. The largest minimum distance between the boundaries of both simplices (see Fig. 1) is given by the distance between their closest corners, i.e., $\varepsilon' = |(1-(\mathcal{N}-1)\varepsilon)-1| + (\mathcal{N}-1)|\varepsilon| = 2(\mathcal{N}-1)\varepsilon$. This is the worst case scenario for $\|b' - b''\|_1$. Then, using the triangular inequality: $\|b - b''\|_1 \leq \|b - b'\|_1 + \|b' - b''\|_1$, and picking the highest possible values for both distances, the bound $\|b - b''\|_1 \leq 2(\mathcal{N}-1)\varepsilon + \delta_B$ holds. □

B.2 Proof of Lemma 4.2

Before proving Lemma 4.2, let us first give an equivalent result in the 1-dimensional case.

Lemma B.1. *Let $x_a, x_b \in \mathbb{R}$ ($x_a < x_b$), and $\eta \in (0, x_b - x_a)$ be three scalars. Let also f be a α-Hölder (with constant K_α), bounded and convex function from $[x_a, x_b]$ to $\mathbb{R}$, f being differentiable everywhere on (x_a, x_b). Then f is $K_\alpha \eta^\alpha$-Lipschitz on $[x_a + \eta, x_b - \eta]$.*

Proof. With any $x \in [x_a + \eta, x_b]$ we have (see Fig. 2 for an illustration):

$$\begin{aligned}
f'(x) &\geq f'(x_a + \eta) && \text{(By convexity of } f\text{)} \\
&\geq \frac{f(x_a+\eta) - f(x_a)}{\eta} && \text{(For the same reason)} \\
&\geq -\frac{|f(x_a+\eta) - f(x_a)|}{\eta} \\
&\geq -K_\alpha \eta^{\alpha-1} && \text{(Because } f \text{ is } \alpha\text{-Hölder).}
\end{aligned}$$

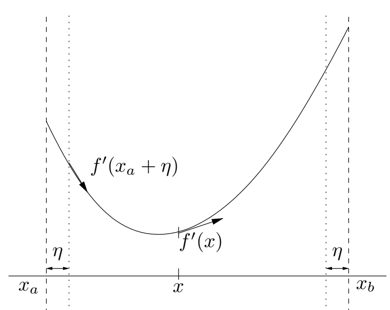

Figure 2: Illustration for the proof of Lemma B.1

Similarly, for any $x \in [x_a, x_b - \eta]$ we have:

$$\begin{aligned}
f'(x) &\leq f'(x_b - \eta) \\
&\leq \frac{f(x_b) - f(x_b - \eta)}{\eta} \\
&\leq \frac{|f(x_b) - f(x_b - \eta)|}{\eta} \\
&\leq K_\alpha \eta^{\alpha-1}.
\end{aligned}$$

Thus, for any $x \in [x_a + \eta, x_b - \eta]$, $|f'(x)| \leq K_\alpha \eta^{\alpha-1}$, so that f is $K_\alpha \eta^{\alpha-1}$-Lipschitz on $[x_a + \eta, x_b - \eta]$. □

We can now show how the above property extends to an n-simplex (using any norm).

Proof of Lemma 4.2. Let b be a point in Δ_η and let $\mathbf{u}$ be a unit vector parallel to the hyperplane containing Δ. As shown on Fig. 3, the line going through b and directed by $\mathbf{u}$ intersects Δ on the closed segment $S_\mathbf{u} = [b + x_a\mathbf{u}, b + x_b\mathbf{u}]$ ($x_a < 0 < x_b$). We can thus define a function g_u from $[x_a, x_b]$ to $\mathbb{R}$ such that $g_\mathbf{u} : x \mapsto f(b + x\mathbf{u})$. $g_\mathbf{u}$ is then a α-Hölder (with same constant), bounded and convex function from $[x_a, x_b]$ to $\mathbb{R}$, $g_\mathbf{u}$ being differentiable everywhere on (x_a, x_b).

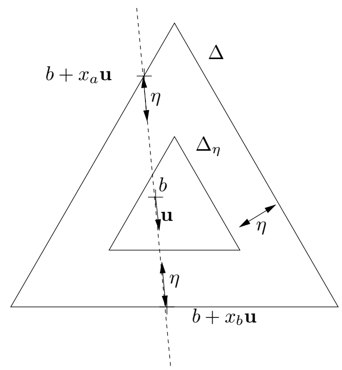

Figure 3: Illustration for the proof of Lemma 4.2

Let us note that the shortest distance (in any norm) between Δ_η and $bd(\Delta)$ (the boundary of Δ) is lower bounded by η.[7] The intersection of Δ_η with $S_\mathbf{u}$ is thus a segment $S'_\mathbf{u} \subseteq [b + (x_a + \eta)\mathbf{u}, b + (x_b - \eta)\mathbf{u}]$. We can therefore apply Lemma B.1 to $g_\mathbf{u}$ in $[x_a + \eta, x_b - \eta]$, what tells us that the derivative of $g_\mathbf{u}$ at point 0 (and therefore the directional derivative of f at b along $\mathbf{u}$) is bounded (in absolute value) by $K_\alpha \eta^{\alpha-1}$. This property holding for any $\mathbf{u}$, we have:

$$\forall b \in \Delta_\eta,\ \|\nabla f(b)\| \le K_\alpha \eta^{\alpha-1}.$$

□

The above lemmas naturally extend to the case where f is piecewise differentiable. Please note that the norm of the gradient of Lemma 4.2 is defined as norm-1 for the uses of this paper, but any p-norm can be used as stated in the proof.

B.3 Proof of Theorem 4.3

Proof. Let us pick $b = \operatorname{argmax}_{x\in\Delta} \epsilon_B(x)$, the point where the approximation ω_B presents the worst error. This value can be bounded as follows:

$$\begin{aligned}
\epsilon_B(b) &\le \rho(b) - \omega_{b^*}(b) && \text{(By Eq. 5 and Convexity)}\\
&\le \rho(b) - \omega_{b''}(b) && \text{($b'' \in B$ makes a worse error)}\\
&= \rho(b) - \rho(b'') + (b'' - b)\cdot\nabla\rho(b'') && \text{(By definition of ω)}\\
&\le |\rho(b) - \rho(b'')| + |(b'' - b)\cdot\nabla\rho(b'')| && \text{(By triangular inequality)}\\
&\le K_\alpha\|b - b''\|_1^\alpha + |(b'' - b)\cdot\nabla\rho(b'')| && \text{(By α-Hölder condition)}\\
&\le K_\alpha\|b - b''\|_1^\alpha + \|\nabla\rho(b'')\|_\infty\|b'' - b\|_1 && \text{(By Hölder inequality)}\\
&\le K_\alpha\|b - b''\|_1^\alpha + \|\nabla\rho(b'')\|_1\|b'' - b\|_1 && \text{(By norm equivalence)}\\
&\le K_\alpha\|b - b''\|_1^\alpha + K_\alpha\eta^{\alpha-1}\|b'' - b\|_1 && \text{(By Lemma 4.2)}\\
&\le K_\alpha(2(\mathcal{N}-1)\varepsilon + \delta_B)^\alpha + K_\alpha\eta^{\alpha-1}(2(\mathcal{N}-1)\varepsilon + \delta_B) && \text{(By Lemma 4.1)}\\
&= K_\alpha((2(\mathcal{N}-1)\varepsilon + \delta_B)^\alpha + (\varepsilon - \delta_B)^{\alpha-1}(2(\mathcal{N}-1)\varepsilon + \delta_B)) && \text{(By definition of η)}
\end{aligned}$$

[7] In p-norm, the distance between Δ_η and $bd(\Delta)$ is the distance between the points $(\eta, \frac{1-\eta}{n-1}, \ldots, \frac{1-\eta}{n-1})$ and $(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$, i.e., $d(\Delta_\eta, bd(\Delta)) = (\eta^p + (n-1)\frac{\eta^p}{(n-1)^p})^{1/p} = \eta(1 + \frac{1}{(n-1)^{p-1}})^{1/p} \ge \eta$.

This last result is a generic bound that depends on the choice of ε, with $\varepsilon \in (\delta_B, \frac{1}{\mathcal{N}}]$. If we define ε as a linear function of δ_B, $\varepsilon = \lambda\delta_B$, then the generic bound can be written as,

$$\epsilon_B(b) \leq K_\alpha \left[(2(\mathcal{N}-1)\lambda+1)^\alpha + (\lambda-1)^{\alpha-1}(2(\mathcal{N}-1)\lambda+1)\right]\delta_B^\alpha = C\delta_B^\alpha \qquad (6)$$

with $\lambda \in (1, \frac{1}{\delta_B \mathcal{N}}]$. $\square$