Supplementary material for "Nash Equilibria of Static Prediction Games"

A Proof of Theorem 2

Proof. Equation 20 defines the Nikaido-Isoda function. Intuitively, $\Psi(\mathbf{a}, \mathbf{b})$ quantifies the sum of the relative cost savings that the players can enjoy by changing from strategy $\mathbf{a}_v$ to strategy $\mathbf{b}_v$ while their opponent continues to play $\mathbf{a}_{-v}$. Equation 21 defines the value function $V(\mathbf{a})$ as the sum of greatest possible cost savings attainable by changing from $\mathbf{a}$ to any strategy unilaterally.

$$\Psi(\mathbf{a}, \mathbf{b}) = \sum_{v \in \{+1,-1\}} \theta_v(\mathbf{a}_v, \mathbf{a}_{-v}) - \theta_v(\mathbf{b}_v, \mathbf{a}_{-v}) \tag{20}$$

$$V(\mathbf{a}) = \max_{\mathbf{b} \in A} \Psi(\mathbf{a}, \mathbf{b}) \tag{21}$$

Equivalently to Equation 2, a Nash equilibrium can be characterized as minimizer of the value function $\mathbf{a}^* = \arg\min_{\mathbf{a}} V(\mathbf{a})$ with $V(\mathbf{a}^*) = 0$. Note, that the value function is always non-negative and that $V(\mathbf{a}^*) = 0$ if and only if $\mathbf{a}^*$ is a Nash equilibrium.

We will now show that V reaches zero at the minimax solution of the joint cost function θ_0. Inserting Equations 15 and 16 into Equation 22 gives Equation 23. Since $\ell_{-1} = -\ell_{+1}$, two of the originally four sums over the examples have canceled each other out. Equation 23 can be written in terms of the joint cost function introduced in Equation 17.

$$\begin{aligned}
\Psi(\mathbf{a}, \mathbf{b}) &= \sum_{v \in \{+1,-1\}} \theta_v(\mathbf{a}_v, \mathbf{a}_{-v}) - \theta_v(\mathbf{b}_v, \mathbf{a}_{-v}) && (22)\\
&= \sum_{i=1}^{n} \ell_{+1}(h_{\mathbf{a}_{+1}}(\phi_{\mathbf{b}_{-1}}(\mathbf{X})_i), y_i) + \Omega_{\mathbf{a}_{+1}} - \Omega_{\mathbf{b}_{-1}} \\
&\quad - \sum_{i=1}^{n} \ell_{+1}(h_{\mathbf{b}_{+1}}(\phi_{\mathbf{a}_{-1}}(\mathbf{X})_i), y_i) - \Omega_{\mathbf{b}_{+1}} + \Omega_{\mathbf{a}_{-1}} && (23)\\
&= \theta_0(\mathbf{a}_{+1}, \mathbf{b}_{-1}) - \theta_0(\mathbf{b}_{+1}, \mathbf{a}_{-1}) && (24)
\end{aligned}$$

A Nash equilibrium is a minimum (over $\mathbf{a}$) of value function V that is in turn the maximum (over $\mathbf{b}$) of the Nikaido-Isoda function Ψ (Equation 25). We then exploit Equation 24. As each of the θ_0 terms depends on two independent pairs of parameters, we can rearrange the max and min expressions as in Equation 28.

$$\begin{aligned}
\Psi\left(\left[\mathbf{a}^*_{+1}, \mathbf{a}^*_{-1}\right], \left[\mathbf{b}^*_{+1}, \mathbf{b}^*_{-1}\right]\right) &= \min_{\mathbf{a}_{+1}, \mathbf{a}_{-1}} \max_{\mathbf{b}_{+1}, \mathbf{b}_{-1}} \Psi([\mathbf{a}_{+1}, \mathbf{a}_{-1}], [\mathbf{b}_{+1}, \mathbf{b}_{-1}]) && (25)\\
&= \min_{\mathbf{a}_{+1}, \mathbf{a}_{-1}} \left(\max_{\mathbf{b}_{-1}} \theta_0(\mathbf{a}_{+1}, \mathbf{b}_{-1}) + \max_{\mathbf{b}_{+1}} -\theta_0(\mathbf{b}_{+1}, \mathbf{a}_{-1}) \right) && (26)\\
&= \min_{\mathbf{a}_{+1}} \max_{\mathbf{b}_{-1}} \theta_0(\mathbf{a}_{+1}, \mathbf{b}_{-1}) + \min_{\mathbf{a}_{-1}} \max_{\mathbf{b}_{+1}} -\theta_0(\mathbf{b}_{+1}, \mathbf{a}_{-1}) && (27)\\
&= \min_{\mathbf{a}_{+1}} \max_{\mathbf{b}_{-1}} \theta_0(\mathbf{a}_{+1}, \mathbf{b}_{-1}) - \max_{\mathbf{a}_{-1}} \min_{\mathbf{b}_{+1}} \theta_0(\mathbf{b}_{+1}, \mathbf{a}_{-1}) && (28)
\end{aligned}$$

As the two summands of Equation 28 are optimization terms over distinct pairs of parameters, two sub-problems can be solved independently:

$$\theta_0(\mathbf{a}^*_{+1}, \mathbf{b}^*_{-1}) = \min_{\mathbf{a}_{+1}} \max_{\mathbf{b}_{-1}} \theta_0(\mathbf{a}_{+1}, \mathbf{b}_{-1}), \tag{29}$$

$$\theta_0(\mathbf{b}^*_{+1}, \mathbf{a}^*_{-1}) = \max_{\mathbf{a}_{-1}} \min_{\mathbf{b}_{+1}} \theta_0(\mathbf{b}_{+1}, \mathbf{a}_{-1}). \tag{30}$$

If $\mathbf{a}^*$ is a Nash equilibrium, then $V(\mathbf{a}^*) = \max_{\mathbf{b}} \Psi(\mathbf{a}^*, \mathbf{b}) = 0$. Any $\mathbf{b}^* = \arg\max_{\mathbf{b}} \Psi(\mathbf{a}^*, \mathbf{b})$ would furthermore constitute an additional Nash equilibrium. Therefore, if the game has a *unique* Nash equilibrium $\mathbf{a}^*$, it follows that $\mathbf{b}^* = \mathbf{a}^*$ and consequently, Equation 29 reduces to $\left[\mathbf{a}^*_{+1}, \mathbf{a}^*_{-1}\right] = \arg\min_{\mathbf{a}_{+1}} \max_{\mathbf{a}_{-1}} \theta_0(\mathbf{a}_{+1}, \mathbf{a}_{-1})$. □